Chapter 1

General Introduction

Instructional Objectives

After reading this chapter the student will be able to

1. Differentiate between various structural forms such as beams, plane truss, space truss, plane frame, space frame, arches, cables, plates and shells.
2. State and use conditions of static equilibrium.
3. Calculate the degree of static and kinematic indeterminacy of a given structure such as beams, truss and frames.
4. Differentiate between stable and unstable structure.
5. Define flexibility and stiffness coefficients.
6. Write force-displacement relations for simple structure.

Introduction

Structural analysis and design is a very old art and is known to human beings since early civilizations. The Pyramids constructed by Egyptians around 2000
B.C. stands today as the testimony to the skills of master builders of that civilization. Many early civilizations produced great builders, skilled craftsmen who constructed magnificent buildings such as the Parthenon at Athens (2500 years old), the great Stupa at Sanchi (2000 years old), Taj Mahal (350 years old), Eiffel Tower (120 years old) and many more buildings around the world. These monuments tell us about the great feats accomplished by these craftsmen in analysis, design and construction of large structures. Today we see around us countless houses, bridges, fly-overs, high-rise buildings and spacious shopping malls. Planning, analysis and construction of these buildings is a science by itself. The main purpose of any structure is to support the loads coming on it by properly transferring them to the foundation. Even animals and trees could be treated as structures. Indeed biomechanics is a branch of mechanics, which concerns with the working of skeleton and muscular structures. In the early periods houses were constructed along the riverbanks using the locally available material. They were designed to withstand rain and moderate wind. Today structures are designed to withstand earthquakes, tsunamis, cyclones and blast loadings. Aircraft structures are designed for more complex aerodynamic loadings. These have been made possible with the advances in structural engineering and a revolution in electronic computation in the past 50 years. The construction material industry has also undergone a revolution in the last four decades resulting in new materials having more strength and stiffness than the traditional construction material.

Here we are mainly concerned with the analysis of framed structures (beam and plane frame), arches, cables and suspension bridges subjected to static loads only. The methods that we would be presenting in this course for analysis of structure were developed based on certain energy principles, which would be discussed here.

Classification of Structures

All structural forms used for load transfer from one point to another are 3- dimensional in nature. In principle one could model them as 3-dimensional elastic structure and obtain solutions (response of structures to loads) by solving the associated partial differential equations. In due course of time, you will appreciate the difficulty associated with the 3-dimensional analysis. Also, in many of the structures, one or two dimensions are smaller than other dimensions. This geometrical feature can be exploited from the analysis point of view. The dimensional reduction will greatly reduce the complexity of associated governing equations from 3 to 2 or even to one dimension. This is indeed at a cost. This reduction is achieved by making certain assumptions (like Bernoulli-Euler' kinematic assumption in the case of beam theory) based on its observed behavior under loads. Structures may be classified as 3-, 2- and 1-dimensional (see Fig. 1.1(a) and (b)). This simplification will yield results of reasonable and acceptable accuracy. Most commonly used structural forms for load transfer are: beams, plane truss, space truss, plane frame, space frame, arches, cables, plates and shells. Each one of these structural arrangement supports load in a specific way.

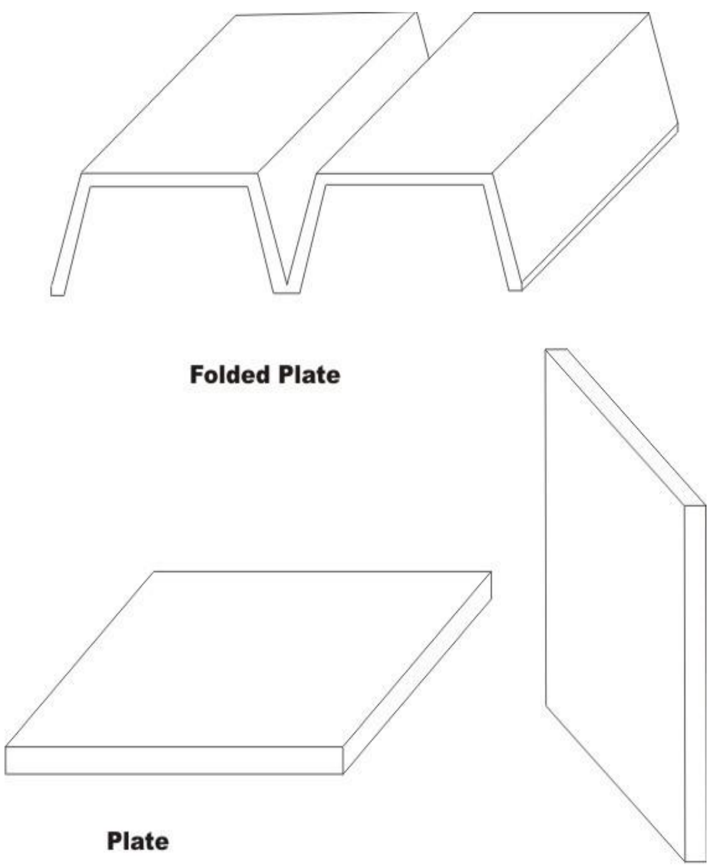
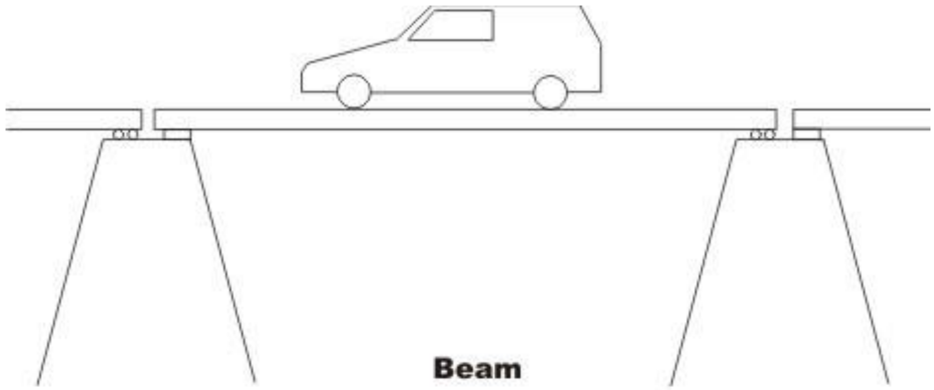
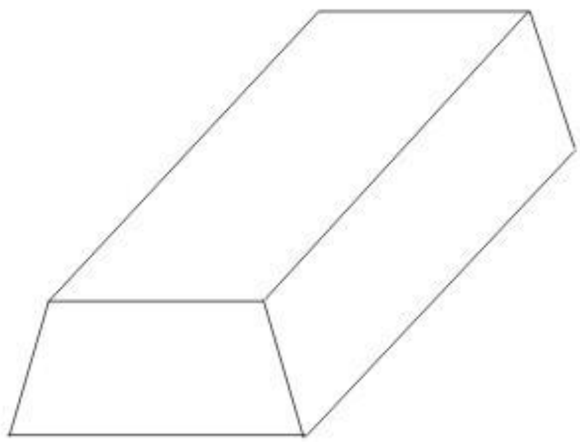


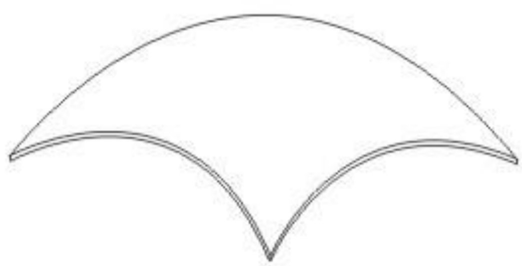
Fig 1.1(a) 2D and 3D Structural Forms



Beam



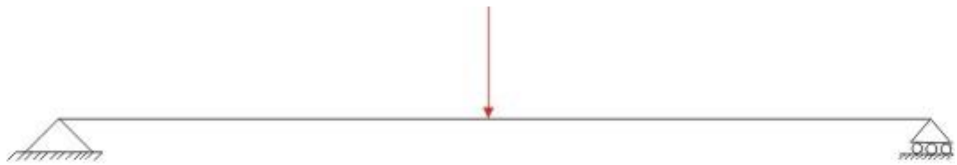
3-D Solid



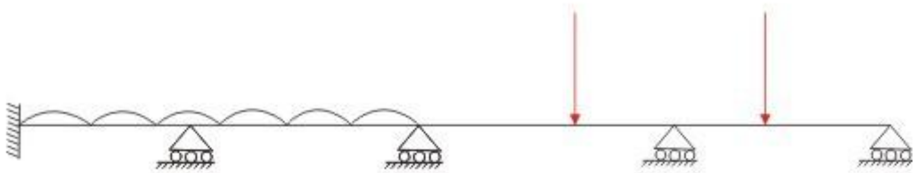
Shell

Fig 1.1(b) Commonly Used Structural Forms

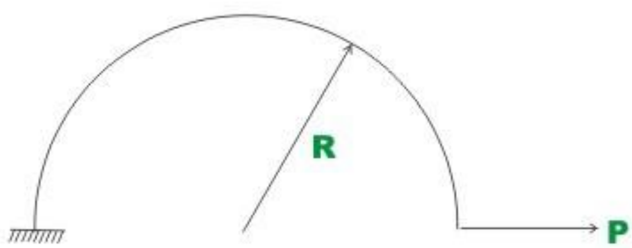
Beams are the simplest structural elements that are used extensively to support loads. They may be straight or curved ones. For example, the one shown in Fig.1.2 (a) is hinged at the left support and is supported on roller at the right end. Usually, the loads are assumed to act on the beam in a plane containing the axis of symmetry of the cross section and the beam axis. The beams may be supported on two or more supports as shown in Fig. 1.2 (b). The beams may be curved in plan as shown in Fig. 1.2 (c). Beams carry loads by deflecting in the same plane and it does not twist. It is possible for the beam to have no axis of symmetry. In such cases, one needs to consider unsymmetrical bending of beams. In general, the internal stresses at any cross section of the beam are: bending moment, shear force and axial force.



(a) Simply Supported Beam



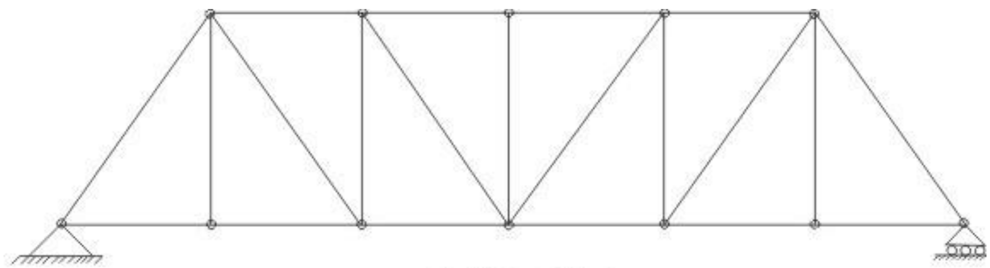
(b) Continuous Beam



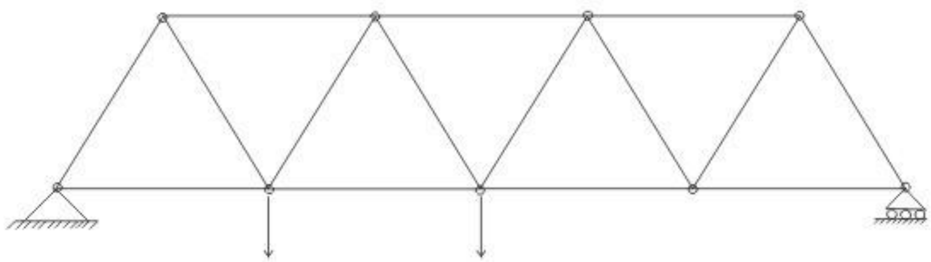
(c) Curved Beam

Fig 1.2 Beams

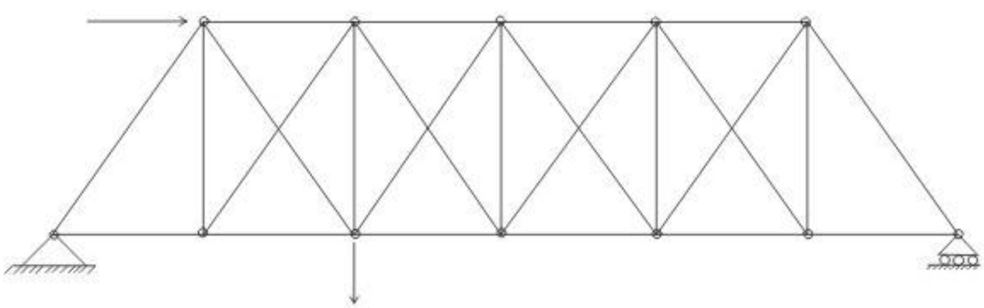
In India, one could see **plane trusses** (vide Fig. 1.3 (a),(b),(c)) commonly in Railway bridges, at railway stations, and factories. Plane trusses are made of short thin members interconnected at hinges into triangulated patterns. For the purpose of analysis statically equivalent loads are applied at joints. From the above definition of truss, it is clear that the members are subjected to only axial forces and they are constant along their length. Also, the truss can have only hinged and roller supports. In field, usually joints are constructed as rigid by welding. However, analyses were carried out as though they were pinned. This is justified as the bending moments introduced due to joint rigidity in trusses are negligible. Truss joint could move either horizontally or vertically or combination of them. In **space truss** (Fig. 1.3 (d)), members may be oriented in any direction. However, members are subjected to only tensile or compressive stresses. Crane is an example of space truss.



(a) Pratt Truss

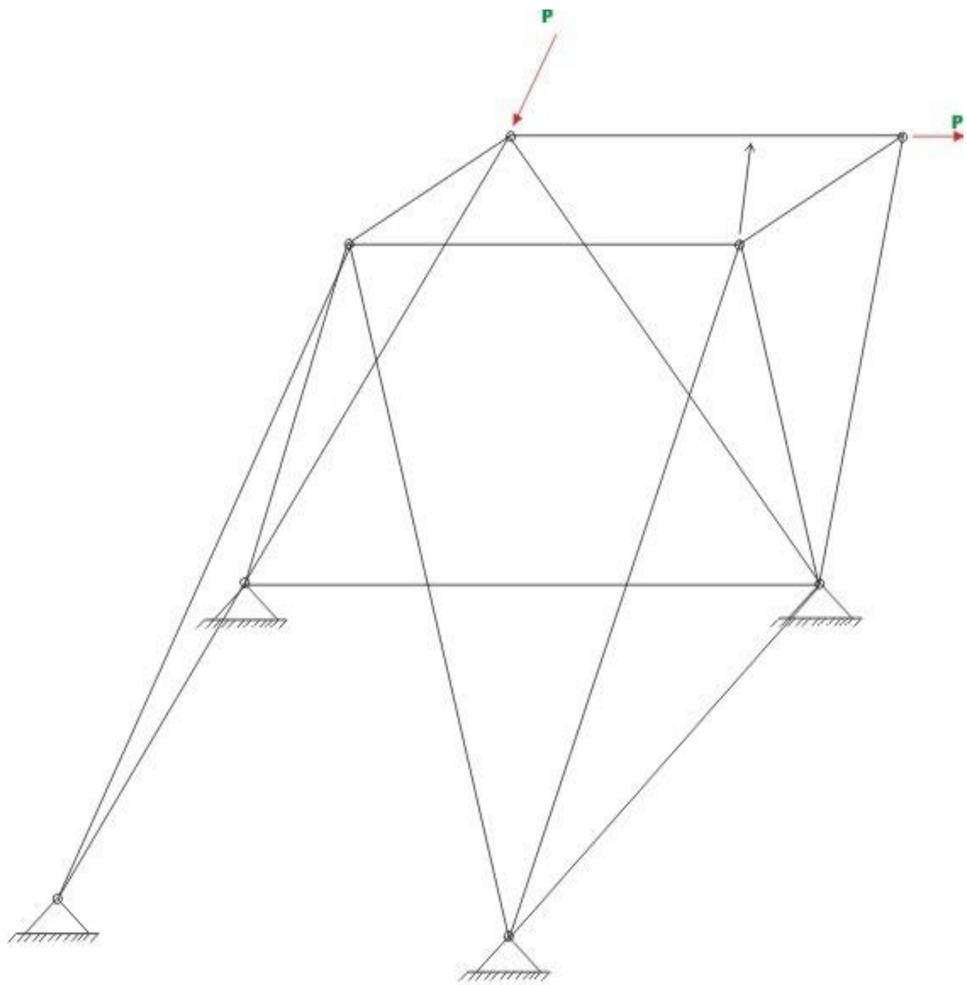


(b) Warren Truss



(c) Double Warren Truss

Fig 1.3 Trusses



(d) Space Truss

Plane frames are also made up of beams and columns, the only difference being they are rigidly connected at the joints as shown in the Fig. 1.4 (a). Major portion of this course is devoted to evaluation of forces in frames for variety of loading conditions. Internal forces at any cross section of the plane frame member are: bending moment, shear force and axial force. As against plane frame, **space frames** (vide Fig. 1.4 (b)) members may be oriented in any direction. In this case, there is no restriction of how loads are applied on the space frame.

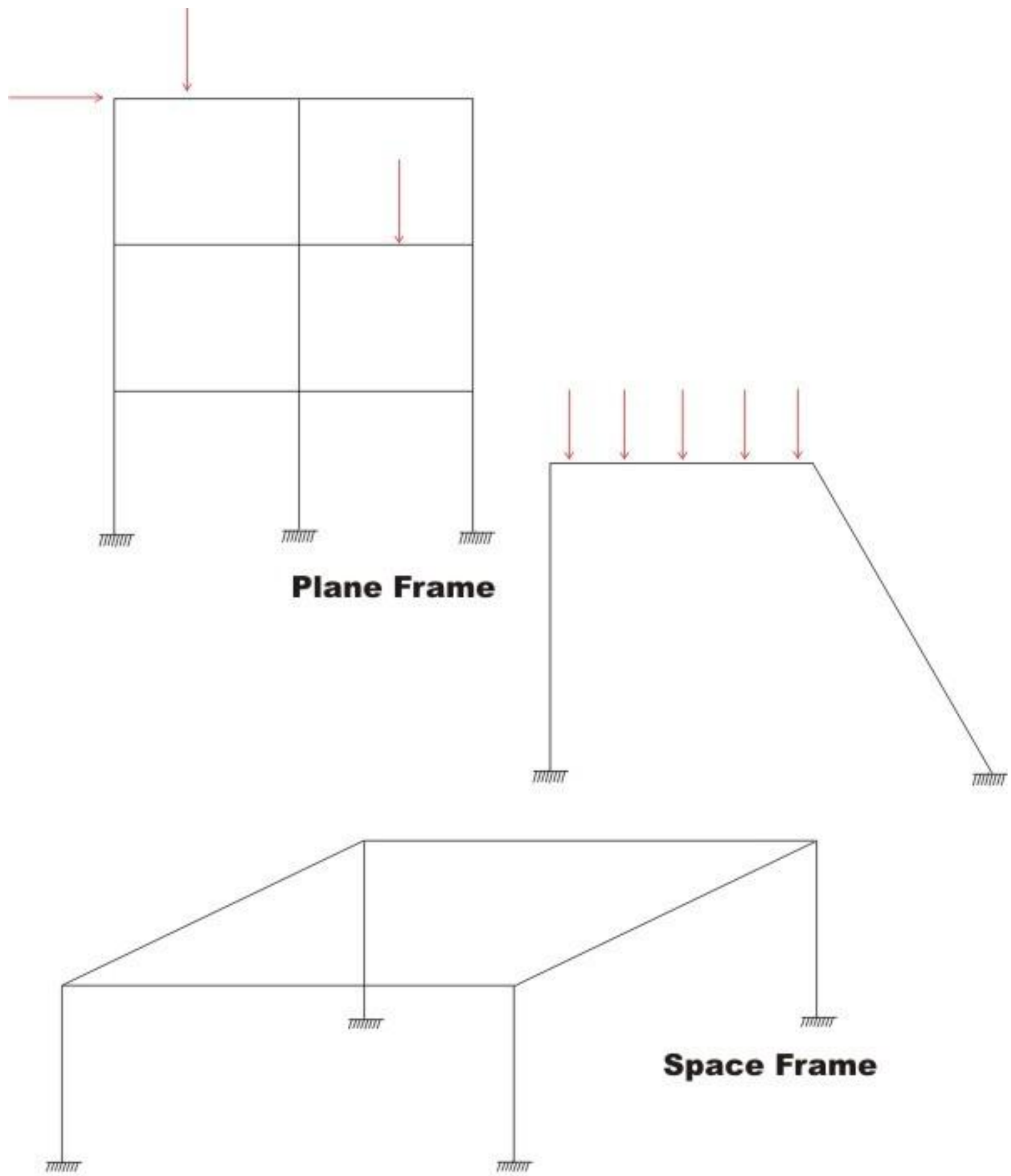


Fig 1.4 Frames

Equations of Static Equilibrium

Consider a case where a book is lying on a frictionless table surface. Now, if we apply a force F_1 horizontally as shown in the Fig.1.5 (a), then it starts moving in the direction of the force. However, if we apply the force perpendicular to the book as in Fig. 1.5 (b), then book stays in the same position, as in this case the vector sum of all the forces acting on the book is zero.

When does an object move and when does it not? This question was answered by Newton when he formulated his famous second law of motion. In a simple vector equation it may be stated as follows:

$$\sum_{i=1}^{n} F_i = ma \qquad (1.1)$$

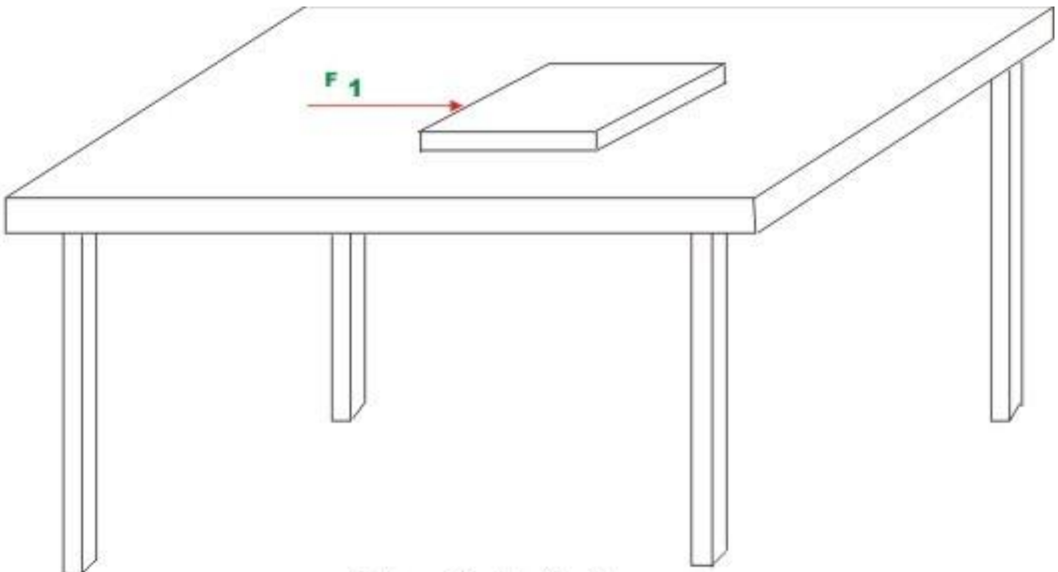


Fig 1.5 (a)

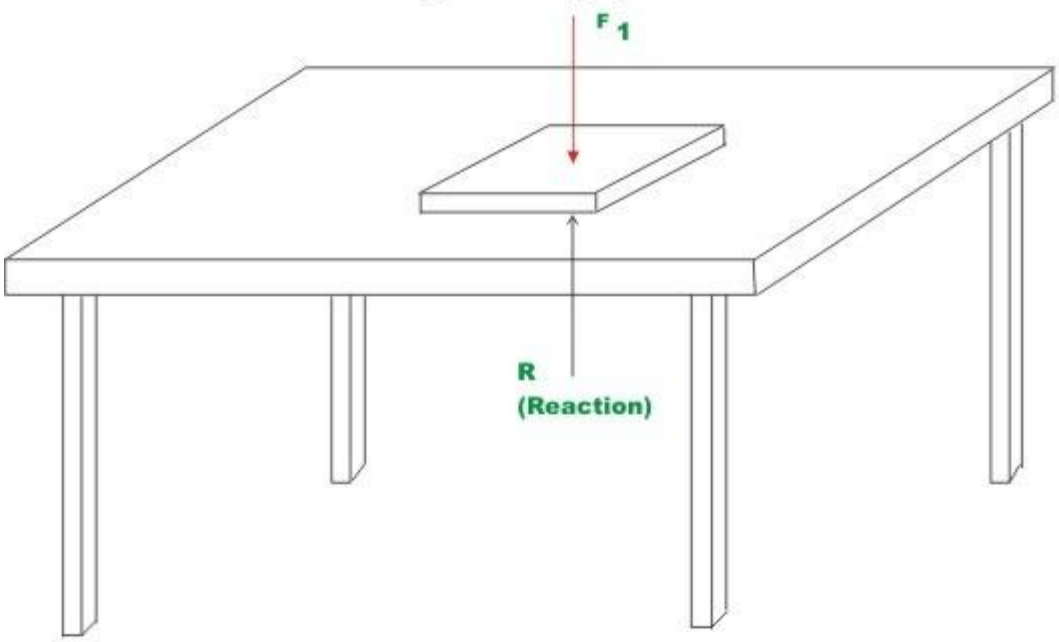


Fig 1.5(b)

where $\sum_{i=1}^{n} F_i$ is the vector sum of all the external forces acting on the body, m is the total mass of the body and a is the acceleration vector. However, if the body is in the state of static equilibrium then the right hand of equation (1.1) must be zero. Also for a body to be in equilibrium, the vector sum of all external moments $(\sum M = 0)$ about an axis through any point within the body must also vanish. Hence, the book lying on the table subjected to external force as shown in Fig. 1.5 (b) is in static equilibrium. The equations of equilibrium are the direct consequences of Newton's second law of motion. A vector in 3-dimensions can be resolved into three orthogonal directions viz., *x, y* and *z* (Cartesian) co- ordinate axes. Also, if the resultant force vector is zero then its components in three mutually perpendicular directions also vanish. Hence, the above two equations may also be written in three co-ordinate axes directions as follows:

$$\sum F_x = 0 \; ; \; \sum F_y = 0 \; ; \; \sum F_z = 0 \tag{1.2a}$$

$$\sum M_x = 0 \; ; \; \sum M_y = 0 \; ; \; \sum M_z = 0 \tag{1.2b}$$

Now, consider planar structures lying in xy plane. For such structures we could have forces acting only in x and y directions. Also the only external moment that could act on the structure would be the one about the z-axis.

For planar structures, the resultant of all forces may be a force, a couple or both. The static equilibrium condition along x-direction requires that there is no net unbalanced force acting along that direction. For such structures we could express equilibrium equations as follows:

$$\sum F_x = 0 \; ; \; \sum F_y = 0 \; ; \; \sum M_z = 0 \tag{1.3}$$

Using the above three equations we could find out the reactions at the supports in the beam shown in Fig. 1.6. After evaluating reactions, one could evaluate internal stress resultants in the beam. Admissible or correct solution for reaction and internal stresses must satisfy the equations of static equilibrium for the entire structure. They must also satisfy equilibrium equations for any part of the structure taken as a free body. If the number of unknown reactions is more than the number of equilibrium equations (as in the case of the beam shown in Fig. 1.7), then we cannot evaluate reactions with only equilibrium equations. Such structures are known as the statically indeterminate structures. In such cases we need to obtain extra equations (compatibility equations) in addition to equilibrium equations.

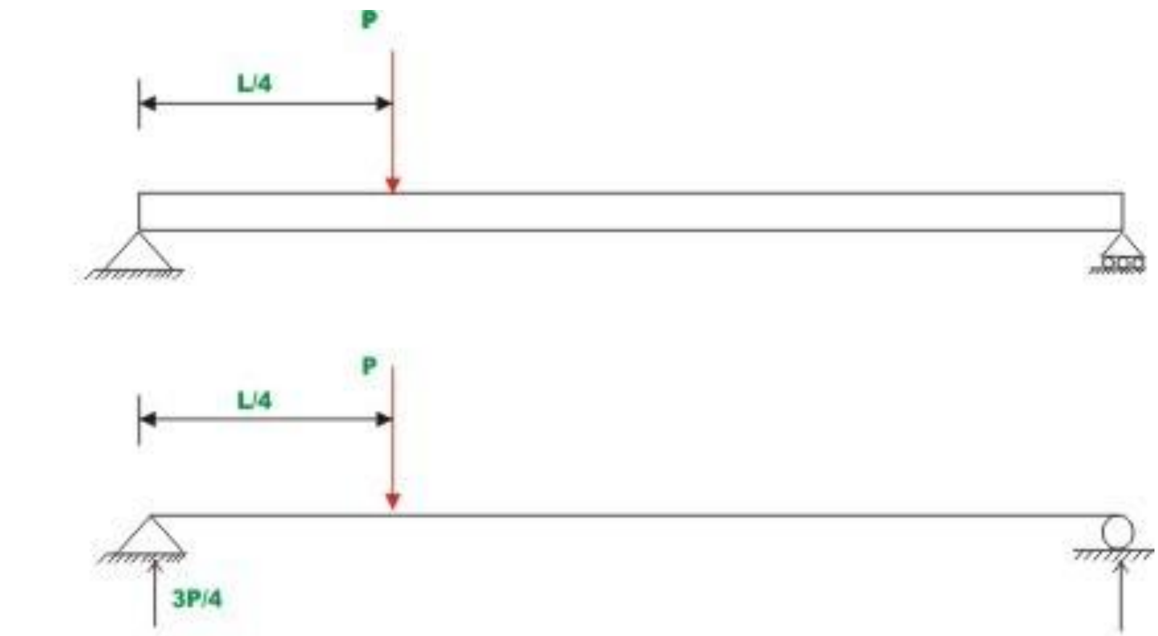


Fig 1.6 Statically Determinate Beam

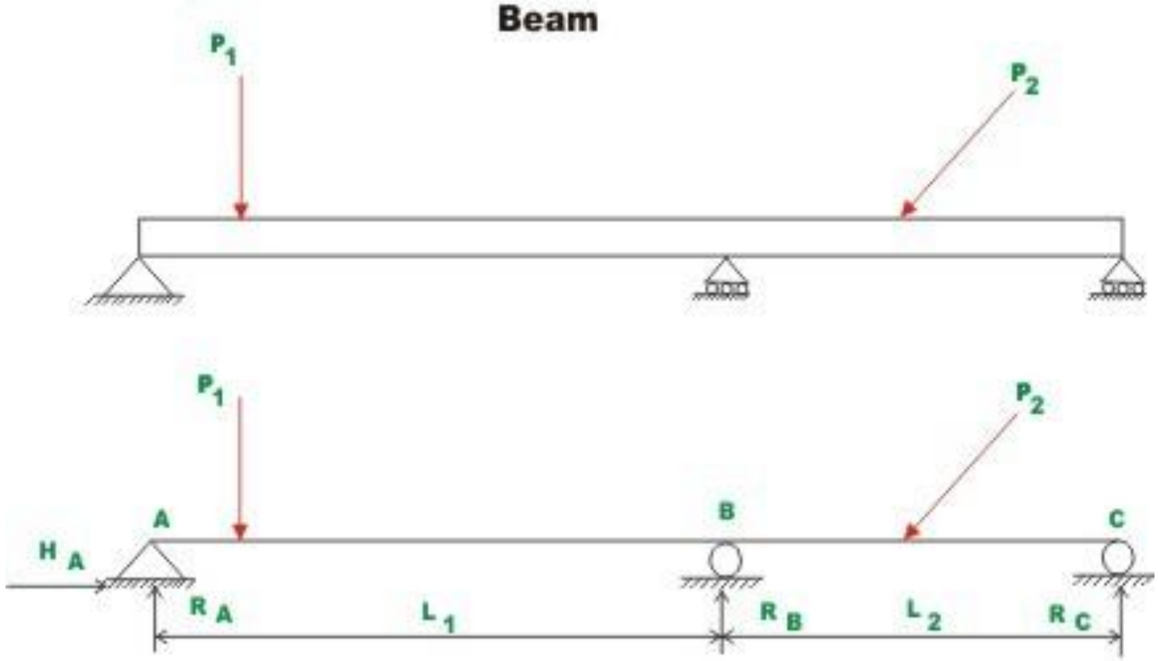


Fig 1.7 Statically Indeterminate Beam

Static Indeterminacy

The aim of structural analysis is to evaluate the external reactions, the deformed shape and internal stresses in the structure. If this can be accomplished by equations of equilibrium, then such structures are known as determinate structures. However, in many structures it is not possible to determine either reactions or internal stresses or both using equilibrium equations alone. Such structures are known as the statically indeterminate structures. The indeterminacy in a structure may be external, internal or both. A structure is said to be externally indeterminate if the number of reactions exceeds the number of equilibrium equations. Beams shown in Fig.1.8(a) and (b) have four reaction components, whereas we have only 3 equations of equilibrium. Hence the beams in Figs. 1.8(a) and (b) are externally indeterminate to the first degree. Similarly, the beam and frame shown in Figs. 1.8(c) and (d) are externally indeterminate to the 3rd degree.

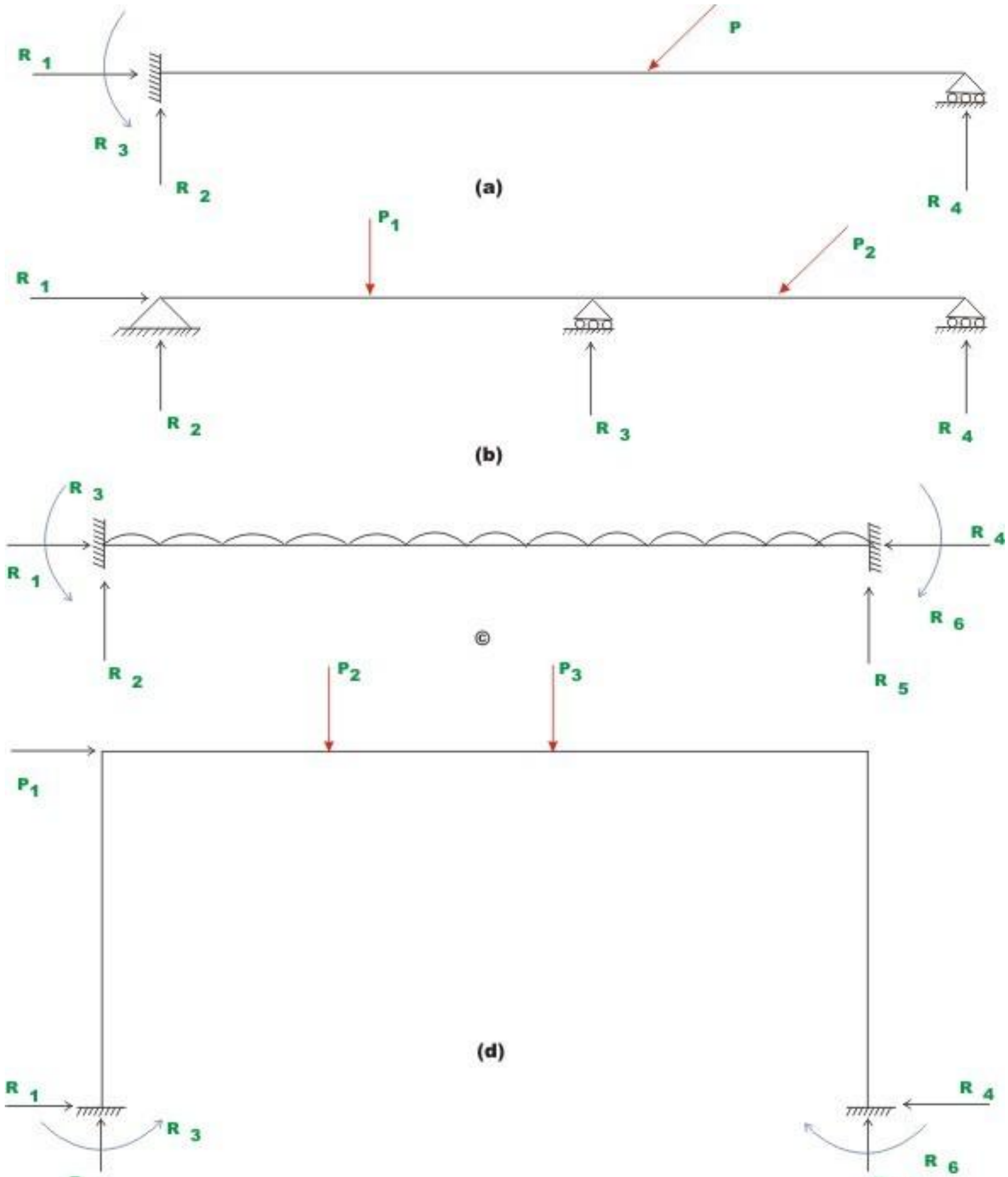
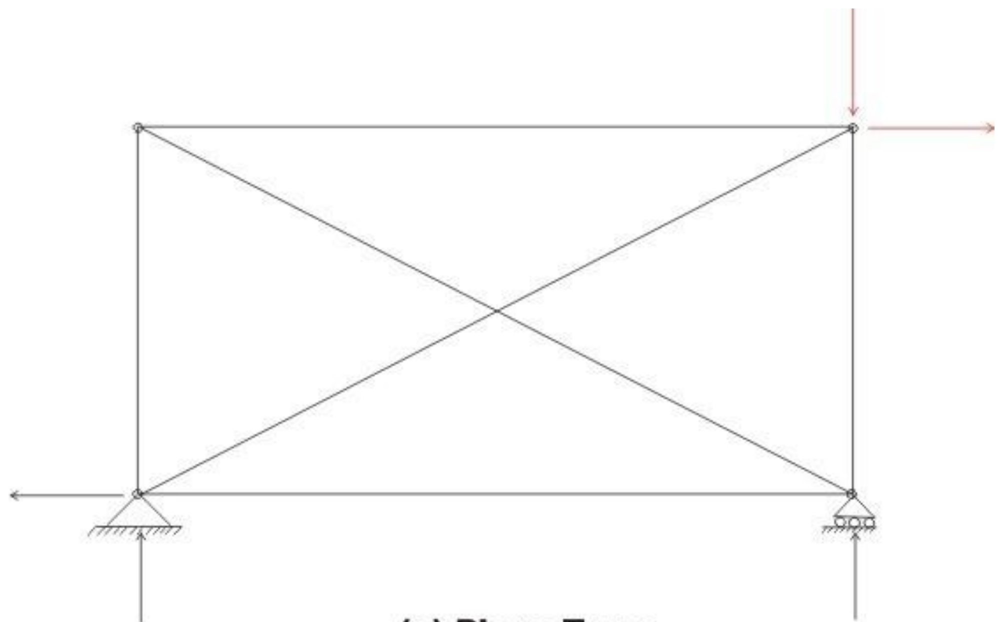
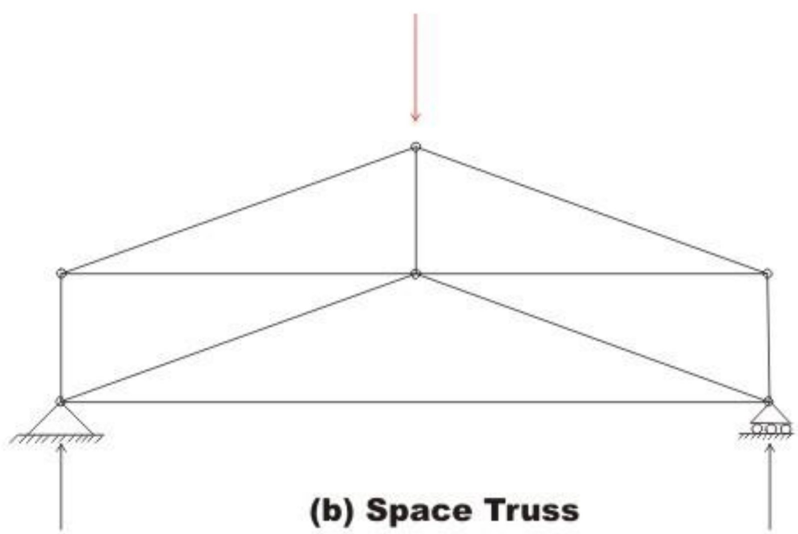


Fig 1.8 Externally Statically Indeterminate Structures

Now, consider trusses shown in Figs. 1.9(a) and (b). In these structures, reactions could be evaluated based on the equations of equilibrium. However, member forces cannot be determined based on statics alone. In Fig. 1.9(a), if one of the diagonal members is removed (cut) from the structure then the forces in the members can be calculated based on equations of equilibrium. Thus, structures shown in Figs. 1.9 (a) and (b) are internally indeterminate to first degree. The truss and frame shown in Fig. 1.10 (a) and (b) are both externally and internally indeterminate.

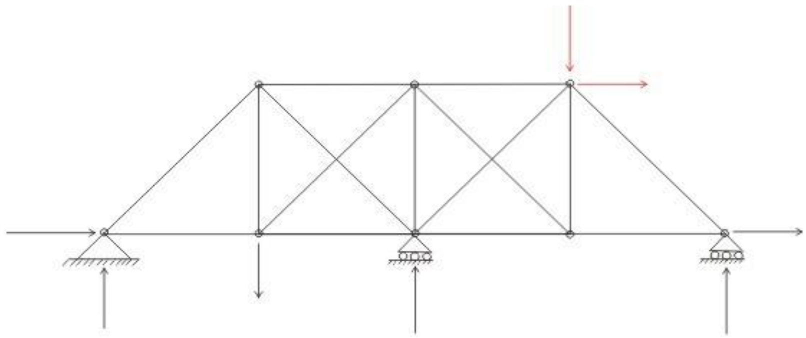


(a) Plane Truss

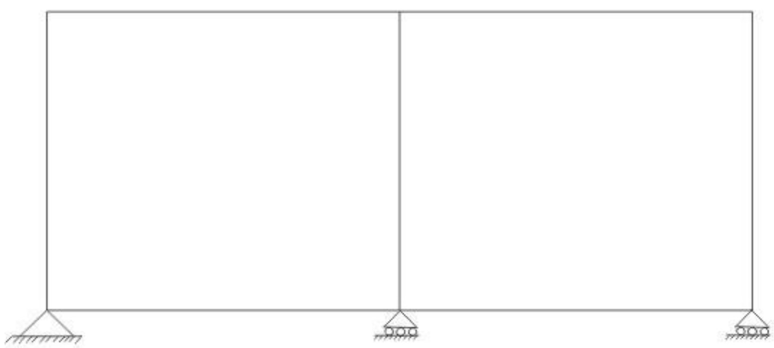


(b) Space Truss

Fig 1.9 Internally Statically Indeterminate Structures



(a) Plane Truss



(b) Plane Frame

Fig 1.10 Externally and Internally Indeterminate Structures

So far, we have determined the degree of indeterminacy by inspection. Such an approach runs into difficulty when the number of members in a structure increases. Hence, let us derive an algebraic expression for calculating degree of static indeterminacy.

Consider a planar stable truss structure having m members and j joints. Let the number of unknown reaction components in the structure be r. Now, the total number of unknowns in the structure is $m + r$. At each joint we could write two equilibrium equations for planar truss structure, viz., $\sum F_x = 0$ and $\sum F_y = 0$. Hence total number of equations that could be written is $2j$.

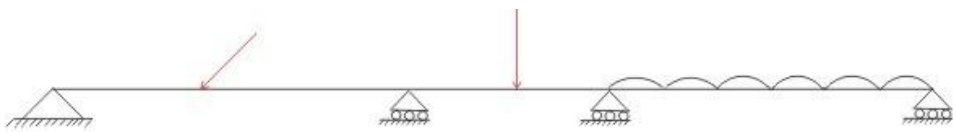
If $2j = m + r$ then the structure is statically determinate as the number of unknowns are equal to the number of equations available to calculate them. The degree of indeterminacy may be calculated as

$$i = (m + r) - 2j \tag{1.4}$$

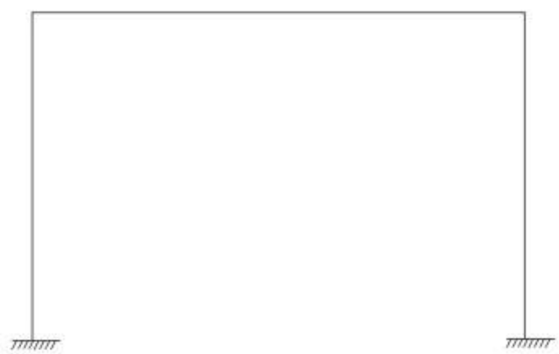
We could write similar expressions for space truss, plane frame, space frame and grillage. For example, the plane frame shown in Fig.1.11 (c) has 15 members, 12 joints and 9 reaction components. Hence, the degree of indeterminacy of the structure is

$$i = (15 \times 3 + 9) - 12 \times 3 = 18$$

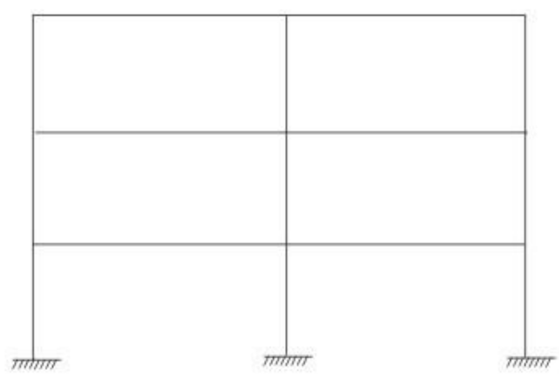
Please note that here, at each joint we could write 3 equations of equilibrium for plane frame.



(a) Continuous Beam



(b) Plane Frame

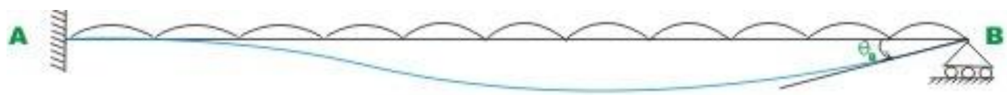


(c) Plane Frame

Fig 1.11 Indeterminate Structures

Kinematic Indeterminacy

When the structure is loaded, the joints undergo displacements in the form of translations and rotations. In the displacement based analysis, these joint displacements are treated as unknown quantities. Consider a propped cantilever beam shown in Fig. 1.12 (a). Usually, the axial rigidity of the beam is so high that the change in its length along axial direction may be neglected. The displacements at a fixed support are zero. Hence, for a propped cantilever beam we have to evaluate only rotation at *B* and this is known as the kinematic indeterminacy of the structure. A fixed beam is kinematically determinate but statically indeterminate to 3rd degree. A simply supported beam and a cantilever beam are kinematically indeterminate to 2nd degree.



(a) Propped Cantilever Beam

(b) Cantilever Beam

(c) Simply Supported Beam

Fig 1.12 Kinematically Indeterminate Structures

The joint displacements in a structure is treated as independent if each displacement (translation and rotation) can be varied arbitrarily and independently of all other displacements. The number of independent joint displacement in a structure is known as the degree of kinematic indeterminacy or the number of degrees of freedom. In the plane frame shown in Fig. 1.13, the joints *B* and *C* have 3 degrees of freedom as shown in the figure. However if axial deformations of the members are neglected then $u_1 = u_4$ and u_2 and u_4 can be neglected. Hence, we have 3 independent joint displacement as shown in Fig. 1.13 i.e. rotations at *B* and *C* and one translation.

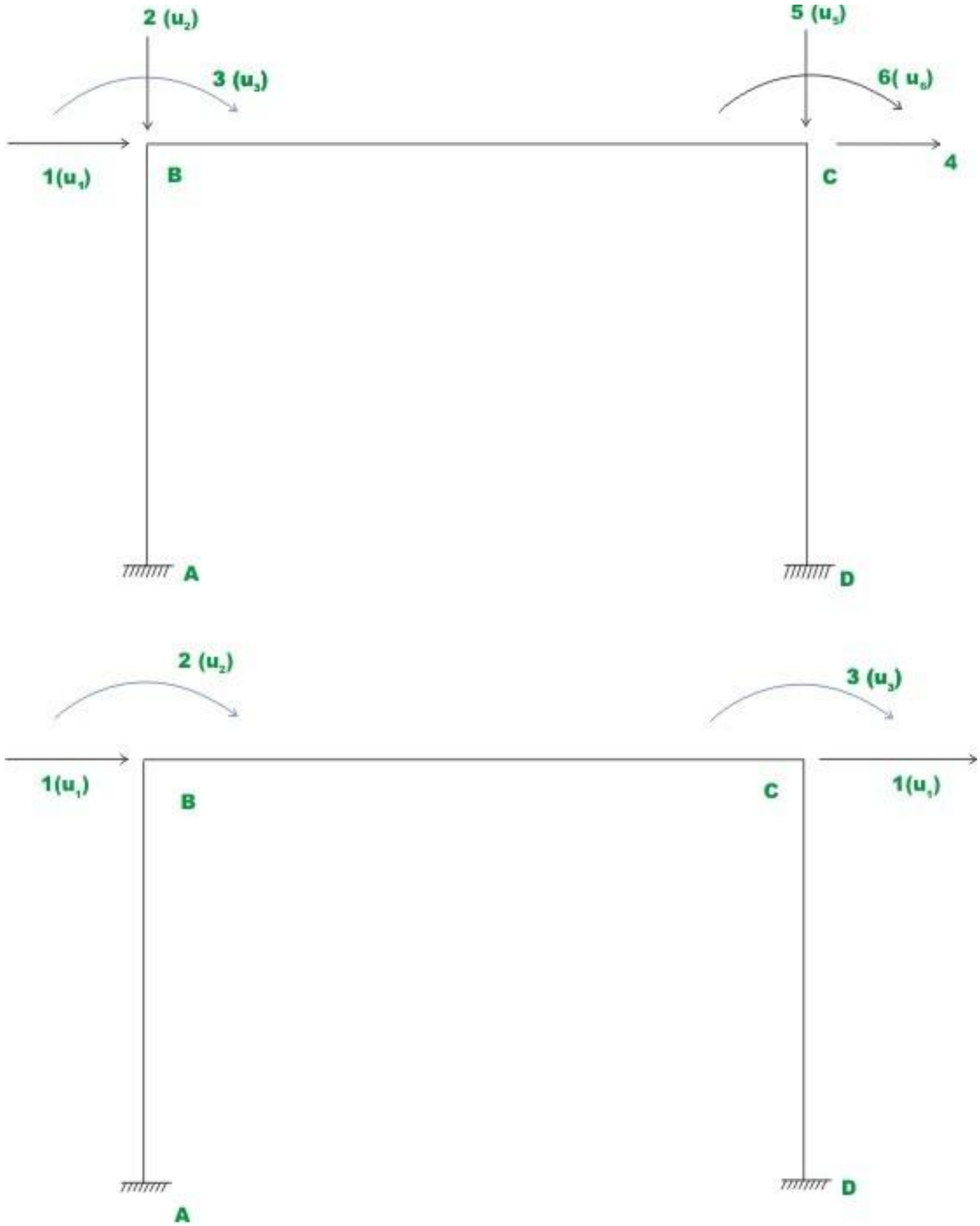


Fig 1.13 Rigid Frame

Kinematically Unstable Structure

A beam which is supported on roller on both ends (Fig. 1.14) on a horizontal surface can be in the state of static equilibrium only if the resultant of the system of applied loads is a vertical force or a couple. Although this beam is stable under special loading conditions, is unstable under a general type of loading conditions. When a system of forces whose resultant has a component in the horizontal direction is applied on this beam, the structure moves as a rigid body. Such structures are known as kinematically unstable structure. One should avoid such support conditions.

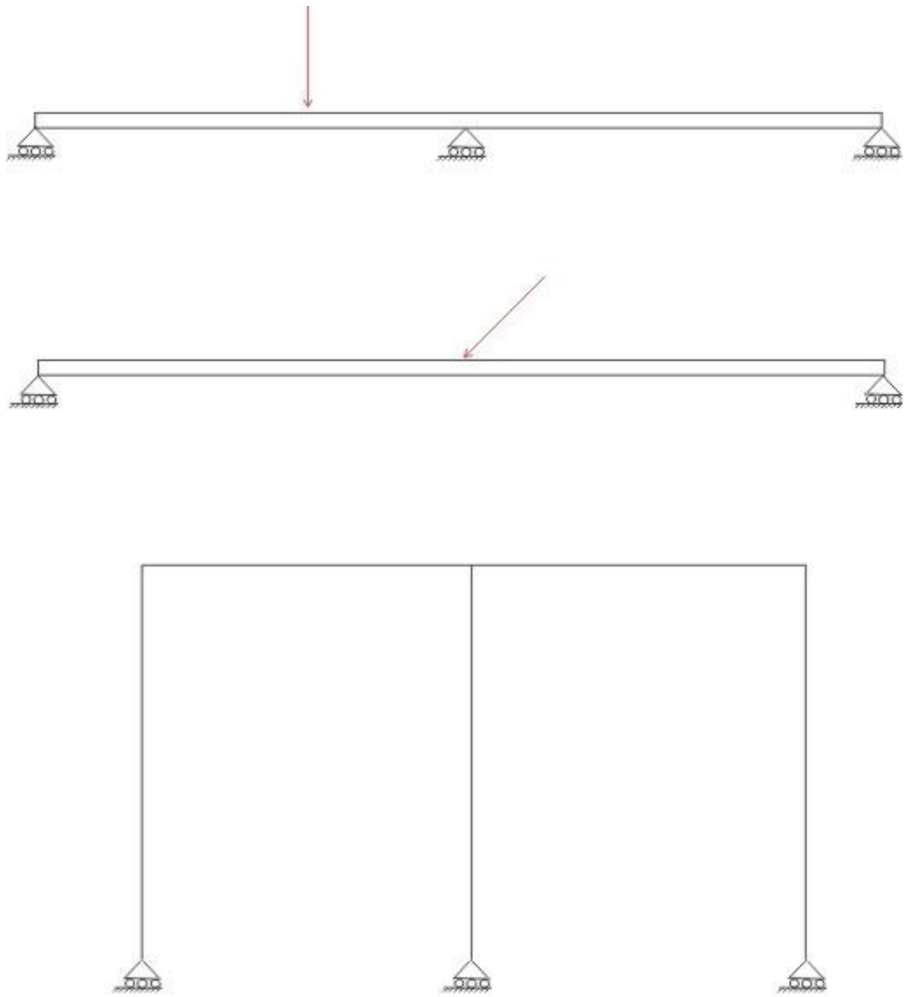


Fig 1.14 Kinematically Unstable Structures

Compatibility Equations

A structure apart from satisfying equilibrium conditions should also satisfy all the compatibility conditions. These conditions require that the displacements and rotations be continuous throughout the structure and compatible with the nature supports conditions. For example, at a fixed support this requires that displacement and slope should be zero.

Force-Displacement Relationship

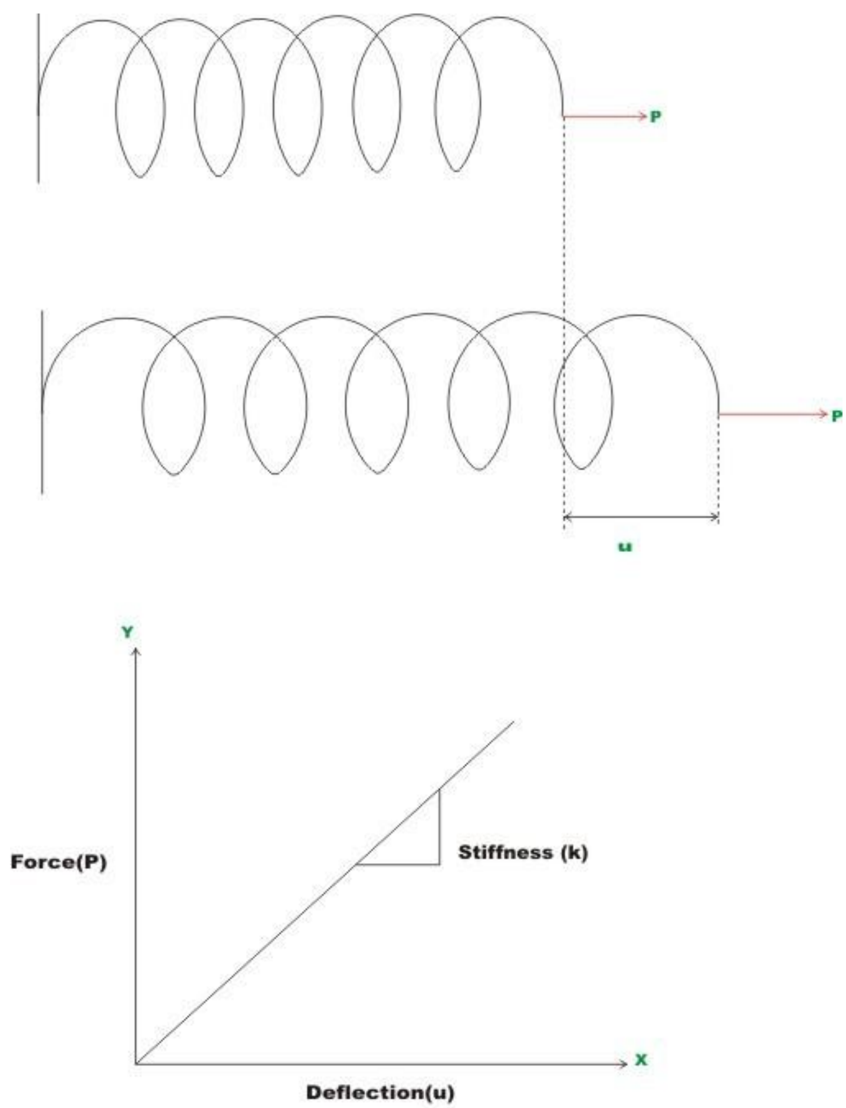


Fig 1.15 Force displacement Relationship

Consider linear elastic spring as shown in Fig.1.15. Let us do a simple experiment. Apply a force P_1 at the end of spring and measure the deformation u_1. Now increase the load to P_2 and measure the deformation u_2. Likewise repeat the experiment for different values of load $P_1, P_2, ..., P_n$. Result may be represented in the form of a graph as shown in the above figure where load is shown on y-axis and deformation on abscissa. The slope of this graph is known as the stiffness of the spring and is represented by k and is given by

$$k = \frac{P_2 - P_1}{u_2 - u_1} = \frac{P}{u} \tag{1.5}$$

$$P = ku \tag{1.6}$$

The spring stiffness may be defined as the force required for the unit deformation of the spring. The stiffness has a unit of force per unit elongation. The inverse of the stiffness is known as flexibility. It is usually denoted by 'a' and it has a unit of displacement per unit force.

$$a = \frac{1}{k} \tag{1.7}$$

the equation (1.6) may be written as

$$P = ku \Rightarrow \qquad u = \frac{1}{k}P = aP \tag{1.8}$$

The above relations discussed for linearly elastic spring will hold good for linearly elastic structures. As an example consider a simply supported beam subjected to a unit concentrated load at the centre. Now the deflection at the centre is given by

$$u = \frac{PL^3}{48EI} \quad \text{or } P = \frac{48EI}{L^3}u \tag{1.9}$$

The stiffness of a structure is defined as the force required for the unit deformation of the structure. Hence, the value of stiffness for the beam is equal to

$$k = \frac{48EI}{L^3}$$

As a second example, consider a cantilever beam subjected to a concentrated load (P) at its tip. Under the action of load, the beam deflects and from first principles the deflection below the load (u) may be calculated as,

$$u = \frac{PL^3}{3EI_{zz}} \tag{1.10}$$

For a given beam of constant cross section, length L , Young's modulus E , and moment of inertia I_{zz} the deflection is directly proportional to the applied load. The equation (1.10) may be written as

$$u = a\,P \tag{1.11}$$

Where 'a is the flexibility coefficient and is $a = \frac{L^3}{3EI_{zz}}$. Usually it is denoted by a_{ij} the flexibility coefficient at i due to unit force applied at j . Hence, the stiffness of beam is

$$k_{11} = \frac{1}{a_{11}} = \frac{3EI}{L^3} \tag{1.12}$$

Summary

In the above discussion we learnt that the structures are classified as: beams, plane truss, space truss, plane frame, space frame, arches, cables, plates and shell depending on how they support external load. The way in which the load is supported by each of these structural systems are discussed. Equations of static equilibrium have been stated with respect to planar and space structures. A brief description of static indeterminacy and kinematic indeterminacy is explained with the help simple structural forms. The kinematically unstable structures are discussed. Compatibility equations and force-displacement relationships are discussed. The term stiffness and flexibility coefficients are defined. The procedure to calculate stiffness of simple structure is discussed.

Suggested Text Books for Further Reading

- Hibbeler, R. C. (2002). *Structural Analysis*, Pearson Education (Singapore) Pte. Ltd., Delhi, ISBN 81-7808-750-2
- Junarkar, S. B. and Shah, H. J. (1999). *Mechanics of Structures – Vol. II*, Charotar Publishing House, Anand.

CHAPTER 2

Principle of Superposition, Strain Energy

Instructional Objectives

After reading this lesson, the student will be able to

1. State and use principle of superposition.
2. Explain strain energy concept.
3. Differentiate between elastic and inelastic strain energy and state units of strain energy.
4. Derive an expression for strain energy stored in one-dimensional structure under axial load.
5. Derive an expression for elastic strain energy stored in a beam in bending.

Introduction

In the analysis of statically indeterminate structures, the knowledge of the displacements of a structure is necessary. Knowledge of displacements is also required in the design of members. Several methods are available for the calculation of displacements of structures. However, if displacements at only a few locations in structures are required then energy based methods are most suitable. If displacements are required to solve statically indeterminate structures, then only the relative values of EA and EI are required. If actual value of displacement is required as in the case of settlement of supports and temperature stress calculations, then it is necessary to know actual values of E . In general deflections are small compared with the dimensions of structure but for clarity the displacements are drawn to a much larger scale than the structure itself. Since, displacements are small, it is assumed not to cause gross displacements of the geometry of the structure so that equilibrium equation can be based on the original configuration of the structure. When non-linear behaviour of the structure is considered then such an assumption is not valid as the structure is appreciably distorted. In this lesson two of the very important concepts i.e., principle of superposition and strain energy method will be introduced.

Principle of Superposition

The principle of superposition is a central concept in the analysis of structures. This is applicable when there exists a linear relationship between external forces and corresponding structural displacements. The principle of superposition may be stated as *the deflection at a given point in a structure produced by several loads acting simultaneously on the structure can be found by superposing deflections at the same point produced by* loads *acting individually*. This is illustrated with the help of a simple beam problem. Now consider a cantilever beam of length L and having constant flexural rigidity EI subjected to two externally applied forces P_1 and P_2 as shown in Fig. 2.1. From moment-area theorem we can evaluate deflection under C , which states that the tangential deviation of point C from the tangent at point A is equal to the first moment of the area of the $\overline{M}$/EI diagram between A and C about C . Hence, the deflection u below C due to loads P_1 and P_2 acting simultaneously is (by moment-area theorem),

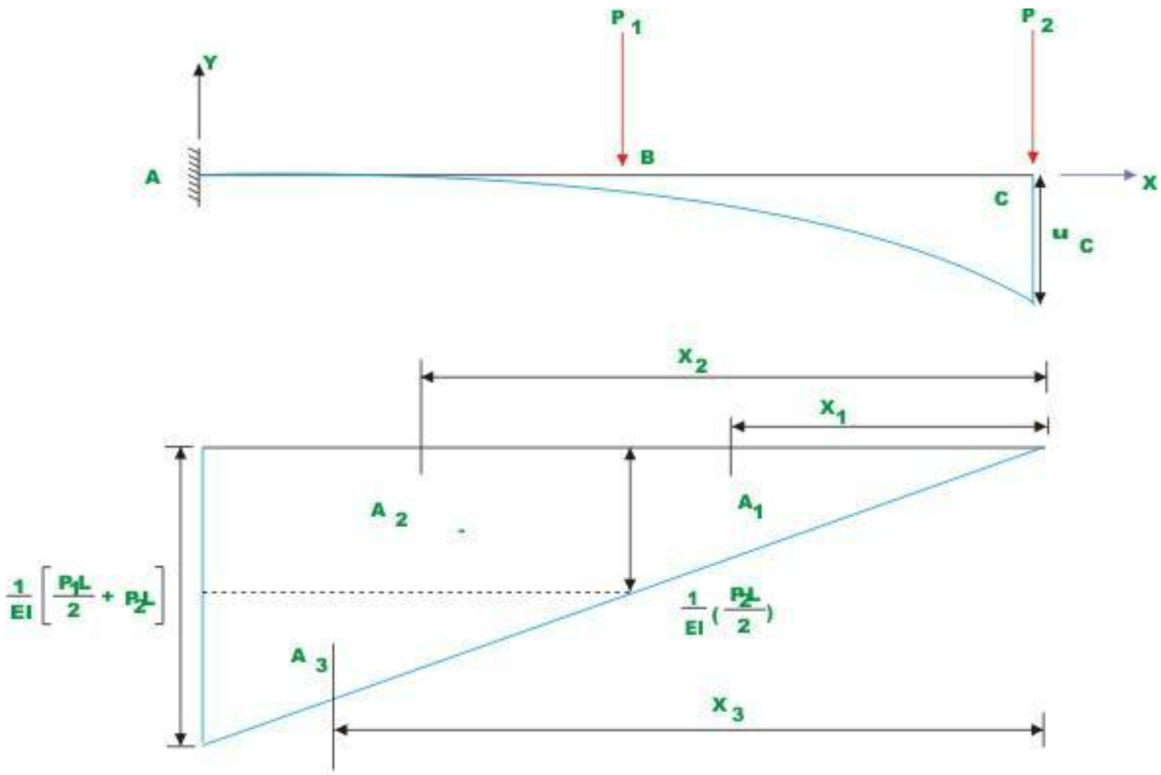


Fig 2.1 Cantilever Beam with Two Concentrated Loads

$$u = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 \qquad (2.1)$$

where u is the tangential deviation of point C with respect to a tangent at A . Since, in this case the tangent at A is horizontal, the tangential deviation of point

C is nothing but the vertical deflection at C. $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are the distances from point C to the centroids of respective areas respectively.

$$\bar{x}_1 = \frac{2}{3}\frac{L}{2} \qquad \bar{x}_2 = \frac{L}{2} + \frac{L}{4} \qquad \bar{x}_3 = \frac{2}{3}\frac{L}{2} + \frac{L}{2}$$

$$A_1 = \frac{P_2 L^2}{8EI} \qquad A_2 = \frac{P_2 L^2}{4EI} \qquad A_3 = \frac{(P_1 L + P_2 L)L}{8EI}$$

Hence,

$$u = \frac{P_2 L^2}{8EI}\frac{2}{3}\frac{L}{2} + \frac{P_2 L^2}{4EI}\left(\frac{L}{2} + \frac{L}{4}\right) + \frac{(P_1 L + P_2 L)L}{8EI}\left(\frac{2}{3}\frac{L}{2} + \frac{L}{2}\right) \tag{2.2}$$

After simplification one can write,

$$u = \frac{P_2 L^3}{3EI} + \frac{5 P_1 L^3}{48EI} \tag{2.3}$$

Now consider the forces being applied separately and evaluate deflection at C in each of the case.

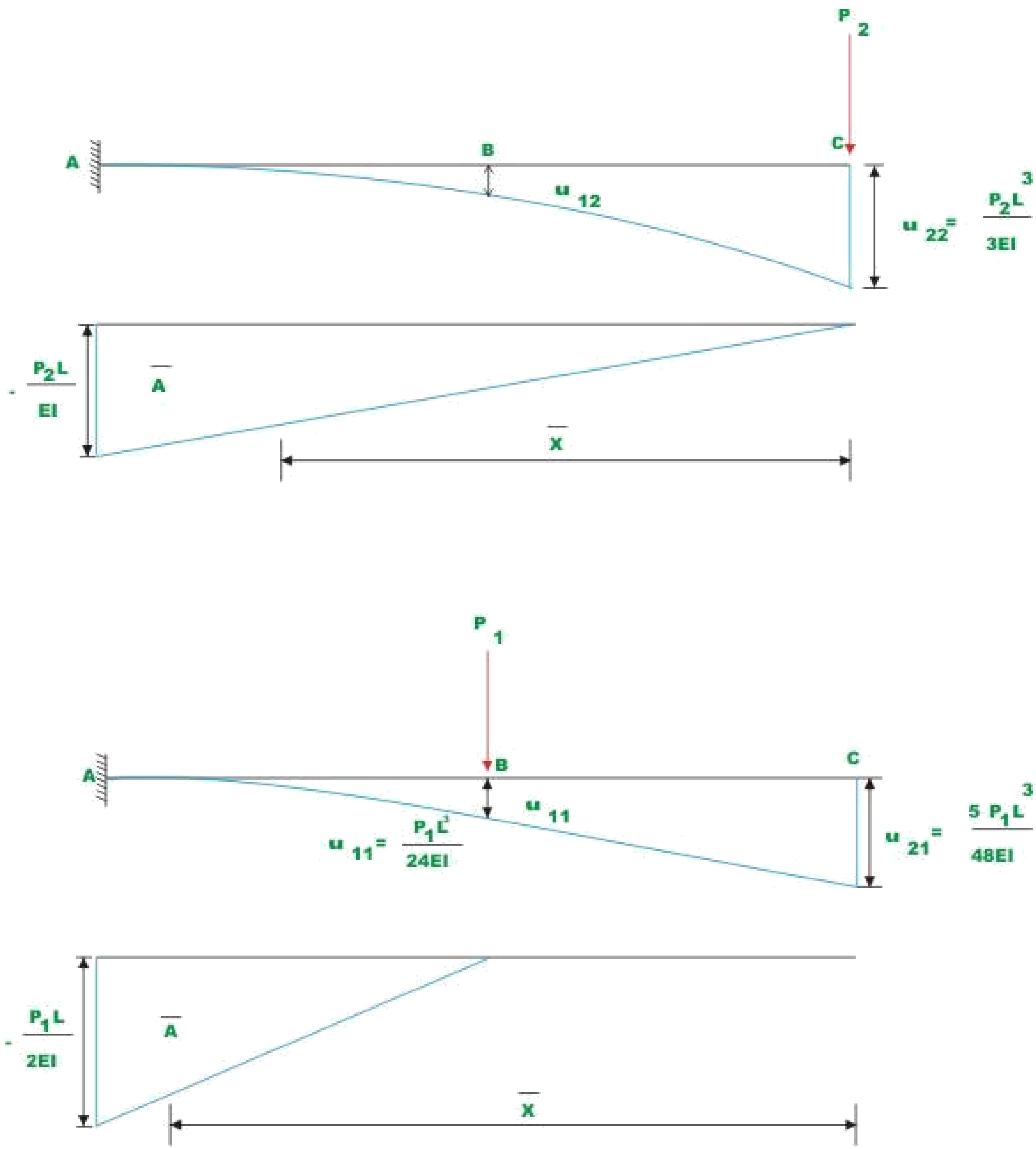


Fig 2.2 Deflection Computation

$$u_{22} = \frac{P_2 L^3}{3EI} \tag{2.4}$$

where u_{22} is deflection at C (2) when load P_1 is applied at C (2) itself. And,

$$u_{21} = \frac{1}{2}\frac{P_1 L}{2EI}\frac{L}{2}\left(\frac{L}{2}+\frac{2}{3}\frac{L}{2}\right) = \frac{5P_1 L^3}{48EI} \tag{2.5}$$

where u_{21} is the deflection at C (2) when load is applied at $B(1)$. Now the total deflection at C when both the loads are applied simultaneously is obtained by adding u_{22} and u_{21}.

$$u = u_{22} + u_{21} = \frac{P_2 L^3}{3EI} + \frac{5P_1 L^3}{48EI} \tag{2.6}$$

Hence it is seen from equations (2.3) and (2.6) that when the structure behaves linearly, the total deflection caused by forces $P_1, P_2, ..., P_n$ at any point in the structure is the sum of deflection caused by forces $P_1, P_2, ..., P_n$ acting independently on the structure at the same point. This is known as the Principle of Superposition.

The method of superposition is not valid when the material stress-strain relationship is non-linear. Also, it is not valid in cases where the geometry of structure changes on application of load. For example, consider a hinged-hinged beam-column subjected to only compressive force as shown in Fig. 2.3(a). Let the compressive force P be less than the Euler's buckling load of the structure. Then deflection at an arbitrary point C (say) u_c^1 is zero. Next, the same beam-column be subjected to lateral load Q with no axial load as shown in Fig. 2.3(b). Let the deflection of the beam-column at C be u_c^2. Now consider the case when the same beam-column is subjected to both axial load P and lateral load Q. As per the principle of superposition, the deflection at the centre u_c^3 must be the sum of deflections caused by P and Q when applied individually. However this is not so in the present case. Because of lateral deflection caused by Q, there will be additional bending moment due to P at C. Hence, the net deflection u_c^3 will be more than the sum of deflections u_c^1 and u_c^2.

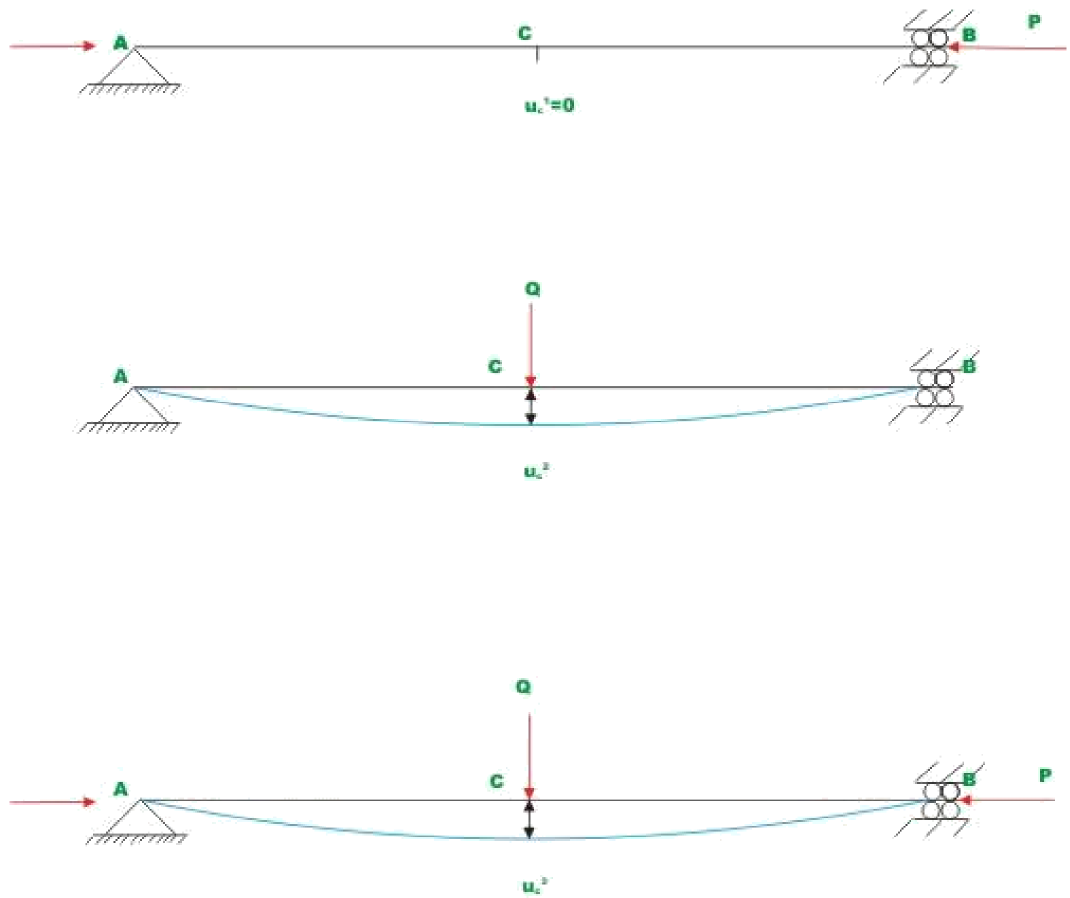


Fig. 2.3

Strain Energy

Consider an elastic spring as shown in the Fig.2.4. When the spring is slowly pulled, it deflects by a small amount u_1 . When the load is removed from the spring, it goes back to the original position. When the spring is pulled by a force, it does some work and this can be calculated once the load-displacement relationship is known. It may be noted that, the spring is a mathematical idealization of the rod being pulled by a force P axially. It is assumed here that the force is applied gradually so that it slowly increases from zero to a maximum value P . Such a load is called static loading, as there are no inertial effects due to motion. Let the load-displacement relationship be as shown in Fig. 2.5. Now, work done by the external force may be calculated as,

$$W_{ext} = \frac{1}{2} P_1 u_1 = \frac{1}{2} (force \times displacement) \quad (2.7)$$

Fig. 2.4 Linear Spring

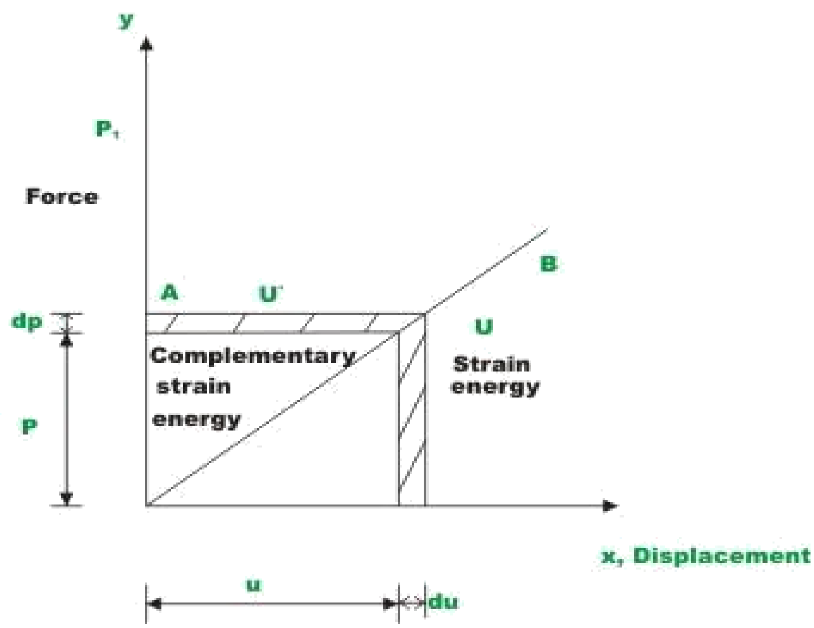


Fig. 2.5 Force-displacement relation

The area enclosed by force-displacement curve gives the total work done by the externally applied load. Here it is assumed that the energy is conserved i.e. the work done by gradually applied loads is equal to energy stored in the structure. This internal energy is known as strain energy. Now strain energy stored in a spring is

$$U = \frac{1}{2} P_1 u_1 \tag{2.8}$$

Work and energy are expressed in the same units. In SI system, the unit of work and energy is joule (J), which is equal to one Newton metre (N.m). The strain energy may also be defined as the internal work done by the stress resultants in moving through the corresponding deformations. Consider an infinitesimal element within a three dimensional homogeneous and isotropic material. In the most general case, the state of stress acting on such an element may be as shown in Fig. 2.6. There are normal stresses $(\sigma_x, \sigma_y \text{ and } \sigma_z)$ and shear stresses $(\tau_{xy}, \tau_{yz} \text{ and } \tau_{zx})$ acting on the element. Corresponding to normal and shear stresses we have normal and shear strains. Now strain energy may be written as,

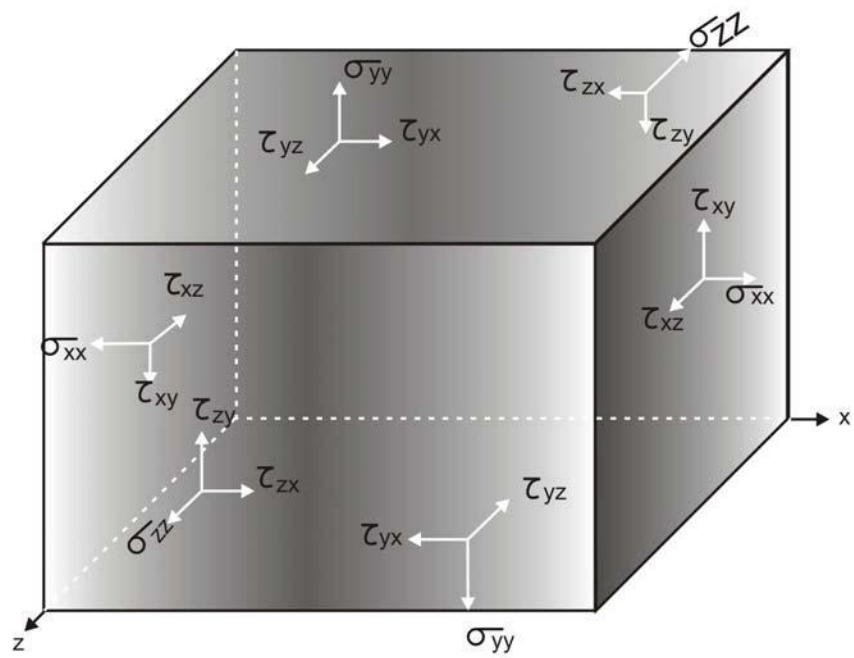


Figure 2.6. Stress on an infinitesimal element .

$$U = \frac{1}{2}\int_v \sigma^T \varepsilon \, dv \tag{2.9}$$

in which σ^T is the transpose of the stress column vector i.e.,

$$\{\sigma\}^T = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}) \text{ and } \{\varepsilon\}^T = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}) \tag{2.10}$$

The strain energy may be further classified as elastic strain energy and inelastic strain energy as shown in Fig. 2.7. If the force P is removed then the spring shortens. When the elastic limit of the spring is not exceeded, then on removal of load, the spring regains its original shape. If the elastic limit of the material is exceeded, a permanent set will remain on removal of load. In the present case, load the spring beyond its elastic limit. Then we obtain the load-displacement curve $OABCDO$ as shown in Fig. 2.7. Now if at B, the load is removed, the spring gradually shortens. However, a permanent set of OD is till retained. The shaded area BCD is known as the elastic strain energy. This can be recovered upon removing the load. The area $OABDO$ represents the inelastic portion of strain energy.

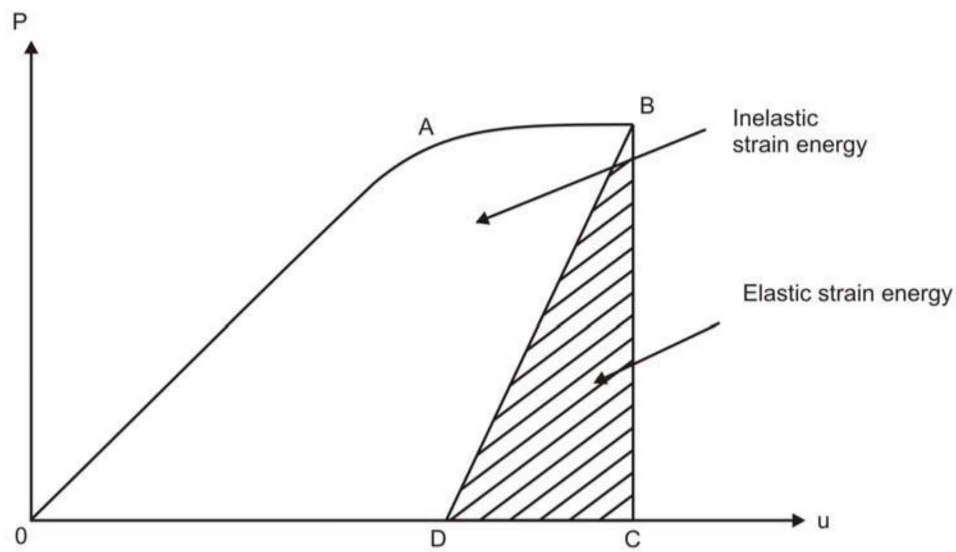


Figure 2.7 Elastic and inelastic strain energy.

The area $OABCDO$ corresponds to strain energy stored in the structure. The area $OABEO$ is defined as the complementary strain energy. For the linearly elastic structure it may be seen that

Area OBC = Area OBE

i.e. Strain energy = Complementary strain energy

This is not the case always as observed from Fig. 2.7. The complementary energy has no physical meaning. The definition is being used for its convenience in structural analysis as will be clear from the subsequent chapters.

Usually structural member is subjected to any one or the combination of bending moment; shear force, axial force and twisting moment. The member resists these external actions by internal stresses. In this section, the internal stresses induced in the structure due to external forces and the associated displacements are calculated for different actions. Knowing internal stresses due to individual forces, one could calculate the resulting stress distribution due to combination of external forces by the method of superposition. After knowing internal stresses and deformations, one could easily evaluate strain energy stored in a simple beam due to axial deformation and , bending deformation.

Strain energy under axial load

Consider a member of constant cross sectional area A , subjected to axial force P as shown in Fig. 2.8. Let E be the Young's modulus of the material. Let the member be under equilibrium under the action of this force, which is applied through the centroid of the cross section. Now, the applied force P is resisted by uniformly distributed internal stresses given by average stress $\sigma = \frac{P}{A}$ as shown by the free body diagram (vide Fig. 2.8). Under the action of axial load P applied at one end gradually, the beam gets elongated by (say) u . This may be calculated as follows. The incremental elongation du of small element of length dx of beam is given by,

$$du = \varepsilon\, dx = \frac{\sigma}{E} dx = \frac{P}{AE} dx \qquad (2.11)$$

Now the total elongation of the member of length L may be obtained by integration

$$u = \int_0^L \frac{P}{AE} dx \qquad (2.12)$$

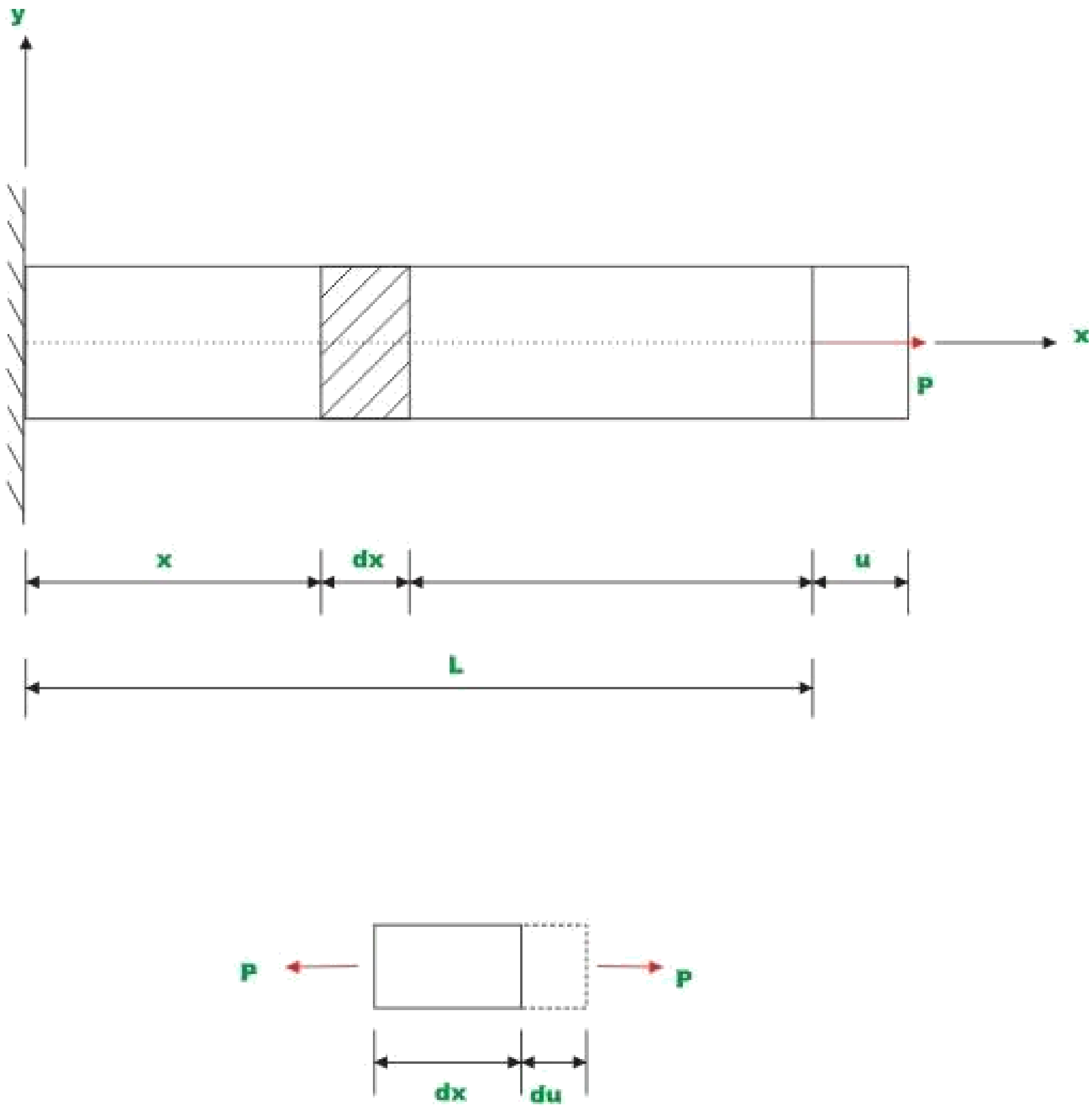


Fig 2.8

Now the work done by external loads $W = \frac{1}{2}Pu$ (2.13)

In a conservative system, the external work is stored as the internal strain energy. Hence, the strain energy stored in the bar in axial deformation is,

$$U = \frac{1}{2}Pu \tag{2.14}$$

Substituting equation (2.12) in (2.14) we get,

$$U = \int_0^L \frac{P^2}{2AE} dx \tag{2.15}$$

Strain energy due to bending

Consider a prismatic beam subjected to loads as shown in the Fig. 2.9. The loads are assumed to act on the beam in a plane containing the axis of symmetry of the cross section and the beam axis. It is assumed that the transverse cross sections (such as AB and CD), which are perpendicular to centroidal axis, remain plane and perpendicular to the centroidal axis of beam (as shown in Fig 2.9).

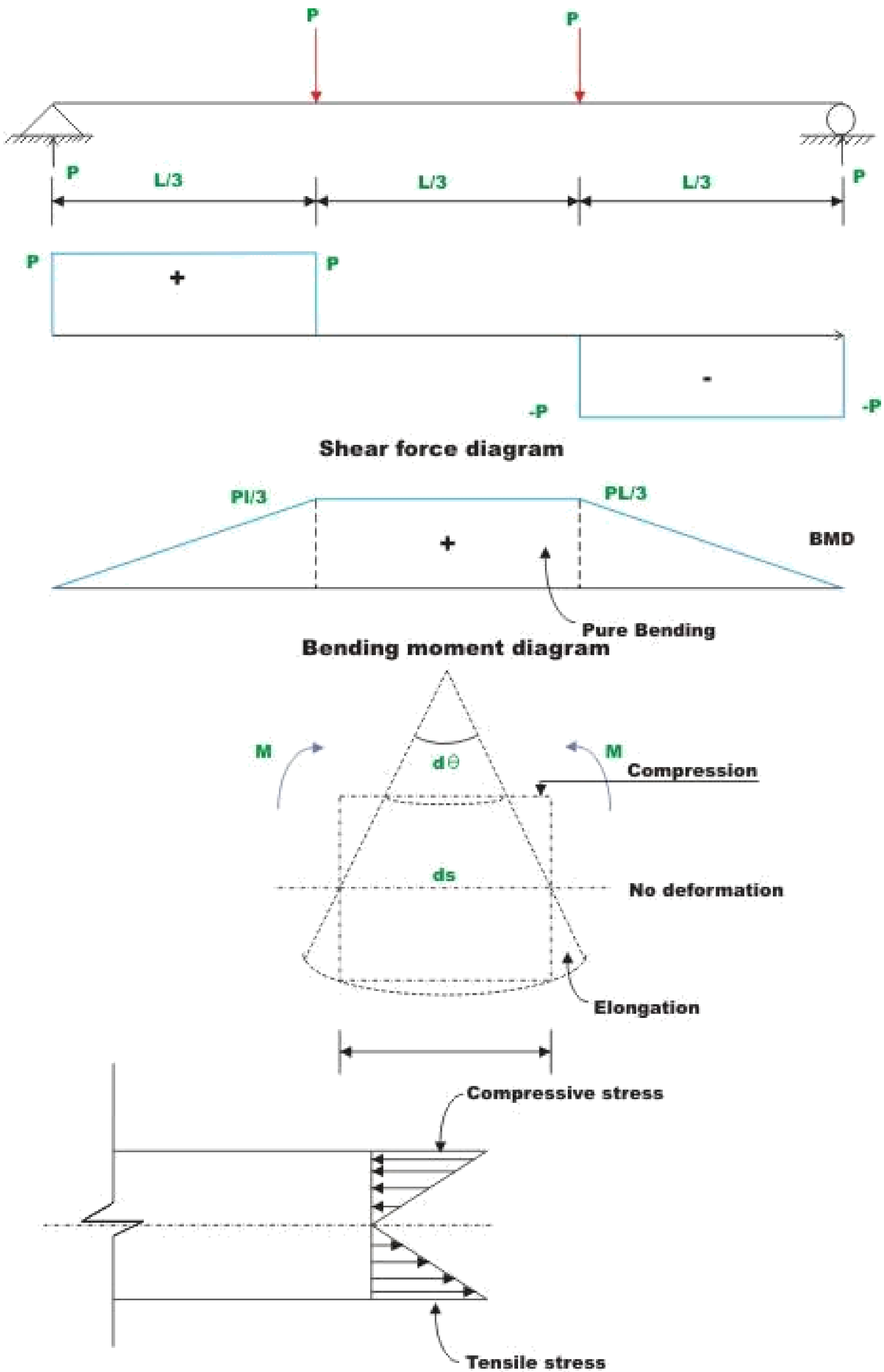


Fig. 2.9 BENDING DEFORMATION

Consider a small segment of beam of length ds subjected to bending moment as shown in the Fig. 2.9. Now one cross section rotates about another cross section by a small amount $d\theta$. From the figure,

$$d\theta = \frac{1}{R} ds = \frac{M}{EI} ds \tag{2.16}$$

where R is the radius of curvature of the bent beam and EI is the flexural rigidity of the beam. Now the work done by the moment M while rotating through angle $d\theta$ will be stored in the segment of beam as strain energy dU. Hence,

$$dU = \frac{1}{2} M \, d\theta \tag{2.17}$$

Substituting for $d\theta$ in equation (2.17), we get,

$$dU = \frac{1}{2} \frac{M^2}{EI} ds \tag{2.18}$$

Now, the energy stored in the complete beam of span L may be obtained by integrating equation (2.18). Thus,

$$U = \int_0^L \frac{M^2}{2EI} ds \tag{2.19}$$

Summary

In this chapter, the principle of superposition has been stated and proved. Also, its limitations have been discussed. It has been shown that the elastic strain energy stored in a structure is equal to the work done by applied loads in deforming the structure. The strain energy expression is also expressed for a 3-dimensional homogeneous and isotropic material in terms of internal stresses and strains in a body. The difference between elastic and inelastic strain energy is explained. Complementary strain energy is discussed. In the end, expressions are derived for calculating strain stored in a simple beam due to axial load and bending moment.

Chapter 3

Castiglione's Theorems

Instructional Objectives

After reading this lesson, the reader will be able to;

1. State and prove first theorem of Castiglione.
2. Calculate deflections along the direction of applied load of a statically determinate structure at the point of application of load.
3. Calculate deflections of a statically determinate structure in any direction at a point where the load is not acting by fictitious (imaginary) load method.
4. State and prove Castiglione’s second theorem.

Introduction

In the previous chapter concepts of strain energy and complementary strain energy were discussed. Castiglione's first theorem is being used in structural analysis for finding deflection of an elastic structure based on strain energy of the structure. The Castiglione's theorem can be applied when the supports of the structure are unyielding and the temperature of the structure is constant.

Castiglione's First Theorem

For linearly elastic structure, where external forces only cause deformations, the complementary energy is equal to the strain energy. For such structures, the Castiglione's first theorem may be stated as the first partial derivative of the strain energy of the structure with respect to any particular force gives the displacement of the point of application of that force in the direction of its line of action.

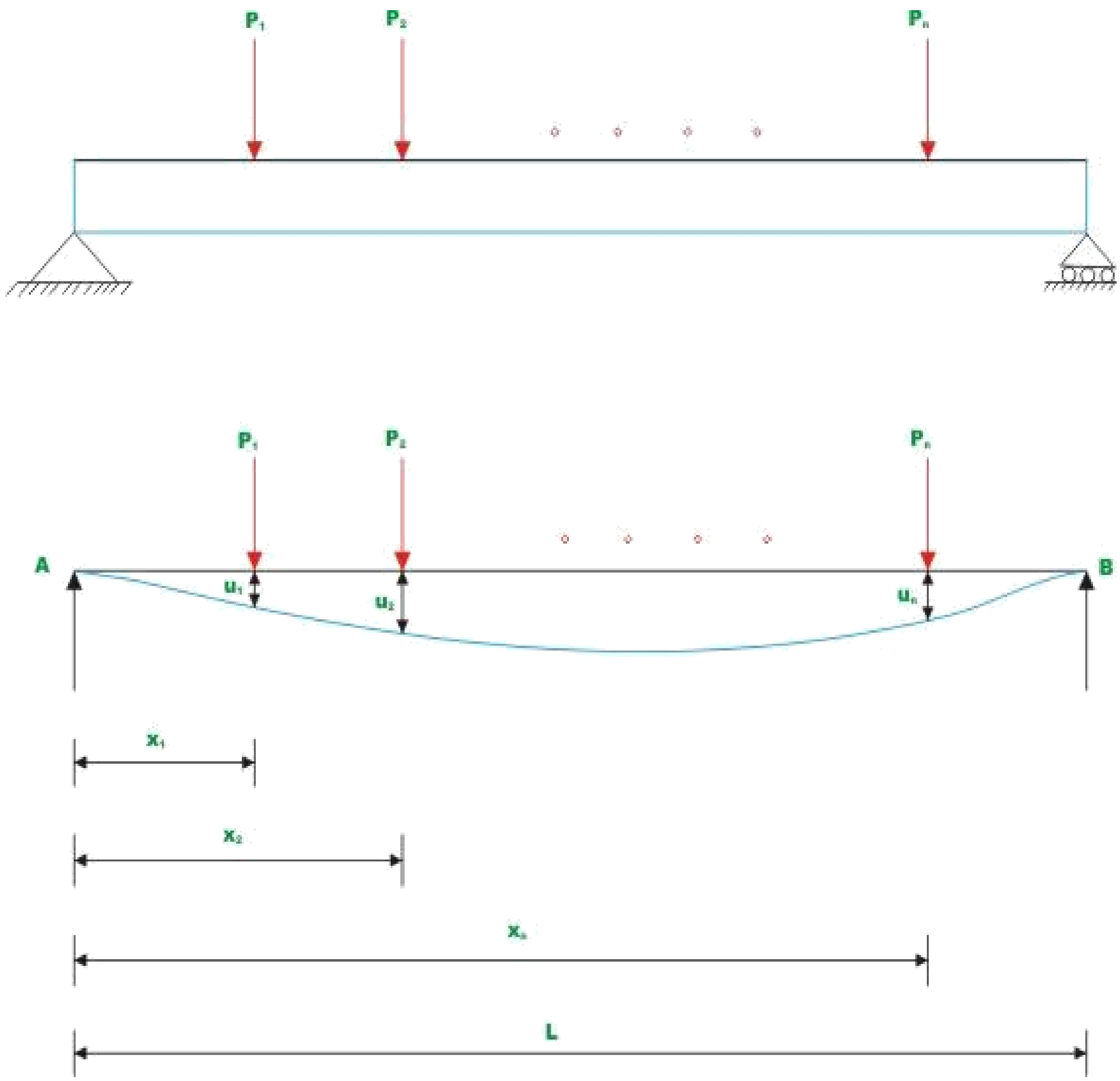


Fig. 3.1 Castigliano's First Theorem

Let P_1 , P_2 ,...., P_n be the forces acting at x_1 , x_2 ,......, x_n from the left end on a simply supported beam of span L . Let u_1 , u_2 ,..., u_n be the displacements at the loading points P_1 , P_2 ,...., P_n respectively as shown in Fig. 3.1. Now, assume that the material obeys Hooke's law and invoking the principle of superposition, the work done by the external forces is given by (vide eqn. 1.8 of lesson 1)

$$W = \frac{1}{2} P_1 u_1 + \frac{1}{2} P_2 u_2 + \ldots\ldots\ldots + \frac{1}{2} P_n u_n \qquad (3.1)$$

Work done by the external forces is stored in the structure as strain energy in a conservative system. Hence, the strain energy of the structure is,

$$U = \frac{1}{2}P_1 u_1 + \frac{1}{2}P_2 u_2 + ... + \frac{1}{2}P_n u_n \tag{3.2}$$

Displacement u_1 below point P_1 is due to the action of $P_1, P_2, ..., P_n$ acting at distances $x_1, x_2, ..., x_n$ respectively from left support. Hence, u_1 may be expressed as,

$$u_1 = a_{11} P_1 + a_{12} P_2 + ... + a_{1n} P_n \tag{3.3}$$

In general,

$$u_i = a_{i1} P_1 + a_{i2} P_2 + ... + a_{in} P_n \qquad i = 1,2,...n \tag{3.4}$$

where a_{ij} is the flexibility coefficient at i due to unit force applied at j. Substituting the values of $u_1, u_2, ..., u_n$ in equation (3.2) from equation (3.4), we get,

$$U = \frac{1}{2} P_1[a_{11} P_1 + a_{12} P_2 + ...] + \frac{1}{2} P_2 [a_{21} P_1 + a_{22} P_2 + ...] + ... + \frac{1}{2} P_n [a_{n1} P_1 + a_{n2} P_2 + ...] \tag{3.5}$$

We know from Maxwell-Betti's reciprocal theorem $a_{ij} = a_{ji}$. Hence, equation (3.5) may be simplified as,

$$U = \frac{1}{2}\left[a_{11} P_1^2 + a_{22} P_2^2 + ... + a_{nn} P_n^2\right] + \left[a_{12} P_1 P_2 + a_{13} P_1 P_3 + ... + a_{1n} P_1 P_n\right] + ... \tag{3.6}$$

Now, differentiating the strain energy with any force P_1 gives,

$$\frac{\partial U}{\partial P_1} = a_{11} P_1 + a_{12} P_2 + ... + a_{1n} P_n \tag{3.7}$$

It may be observed that equation (3.7) is nothing but displacement u_1 at the loading point.

In general,

$$\frac{\partial U}{\partial P_n} = u_n \tag{3.8}$$

Hence, for determinate structure within linear elastic range the partial derivative of the total strain energy with respect to any external load is equal to the displacement of the point of application of load in the direction of the applied load, provided the supports are unyielding and temperature is maintained constant. This theorem is advantageously used for calculating deflections in elastic structure. The procedure for calculating the deflection is illustrated with few examples.

Example 3.1

Find the displacement and slope at the tip of a cantilever beam loaded as in Fig. 3.2. Assume the flexural rigidity of the beam EI to be constant for the beam.

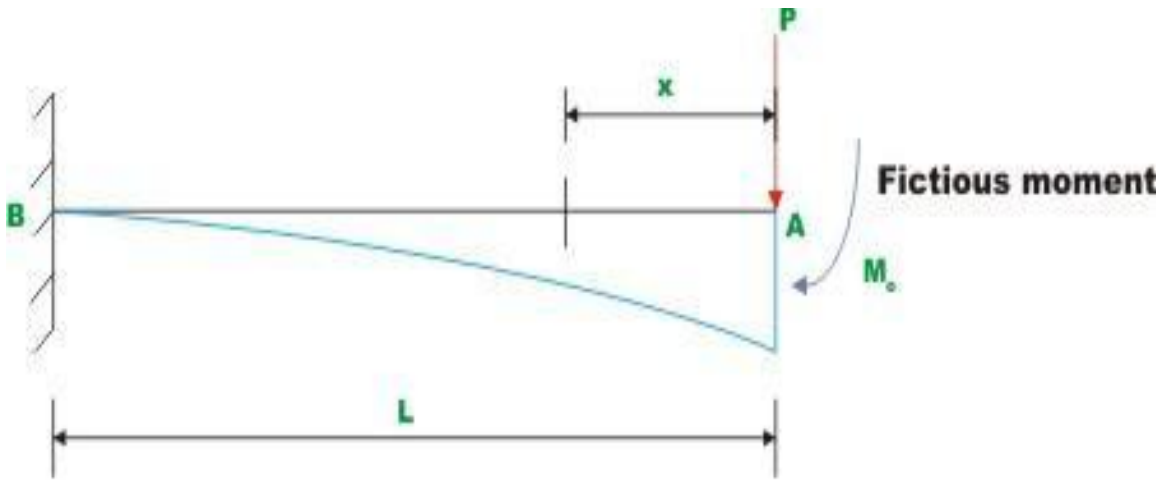


Fig. 3.2 Example 3.1

Moment at any section at a distance x away from the free end is given by

$$M = -Px \tag{1}$$

Strain energy stored in the beam due to bending is $U = \int_0^L \frac{M^2}{2EI} dx$ (2)

Substituting the expression for bending moment M in equation (3.10), we get,

$$U = \int_0^L \frac{(Px)^2}{2EI} dx = \frac{P^2 L^3}{6EI} \tag{3}$$

Now, according to Castigliano's theorem, the first partial derivative of strain energy with respect to external force P gives the deflection u_A at A in the direction of applied force. Thus,

$$\frac{\partial U}{\partial P} = u_A = \frac{PL^3}{3EI} \tag{4}$$

To find the slope at the free end, we need to differentiate strain energy with respect to externally applied moment M at A. As there is no moment at A, apply a fictitious moment M_0 at A. Now moment at any section at a distance x away from the free end is given by

$$M = -Px - M_0$$

Now, strain energy stored in the beam may be calculated as,

$$U = \int_0^L \frac{(Px + M_0)^2}{2EI} dx = \frac{P^2 L^3}{6EI} + \frac{M_0 PL^2}{2EI} + \frac{M_0^2 L}{2EI} \tag{5}$$

Taking partial derivative of strain energy with respect to M_0, we get slope at A.

$$\frac{\partial U}{\partial M_0} = \theta_A = \frac{PL^2}{2EI} + \frac{M_0 L}{EI} \tag{6}$$

But actually there is no moment applied at A. Hence substitute $M_0 = 0$ in equation (3.14) we get the slope at A.

$$\theta_A = \frac{PL^2}{2EI} \tag{7}$$

Example 3.2

A cantilever beam which is curved in the shape of a quadrant of a circle is loaded as shown in Fig. 3.3. The radius of curvature of curved beam is R, Young's modulus of the material is E and second moment of the area is I about an axis perpendicular to the plane of the paper through the centroid of the cross section. Find the vertical displacement of point A on the curved beam.

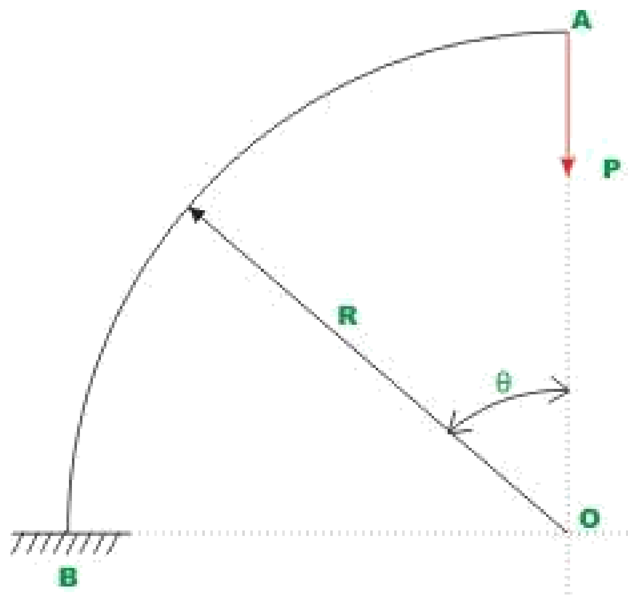


Fig. 3.3 Example 3.2

The bending moment at any section θ of the curved beam (see Fig. 3.3) is given by

$$M = PR\sin\theta \qquad (1)$$

Strain energy U stored in the curved beam due to bending is,

$$U = \int_0^s \frac{M^2}{2EI}\,ds = \int_0^{\pi/2} \frac{P^2R^2(\sin^2\theta)R\,d\theta}{2EI} = \frac{P^2R^3}{2EI}\frac{\pi}{4} = \frac{\pi P^2R^3}{8EI} \qquad (2)$$

Differentiating strain energy with respect to externally applied load, P we get

$$u_A = \frac{\partial U_b}{\partial P} = \frac{\pi PR^3}{4EI} \qquad (3)$$

Example 3.3

Find horizontal displacement at D of the frame shown in Fig. 3.4. Assume the flexural rigidity of the beam EI to be constant through out the member. Neglect strain energy due to axial deformations.

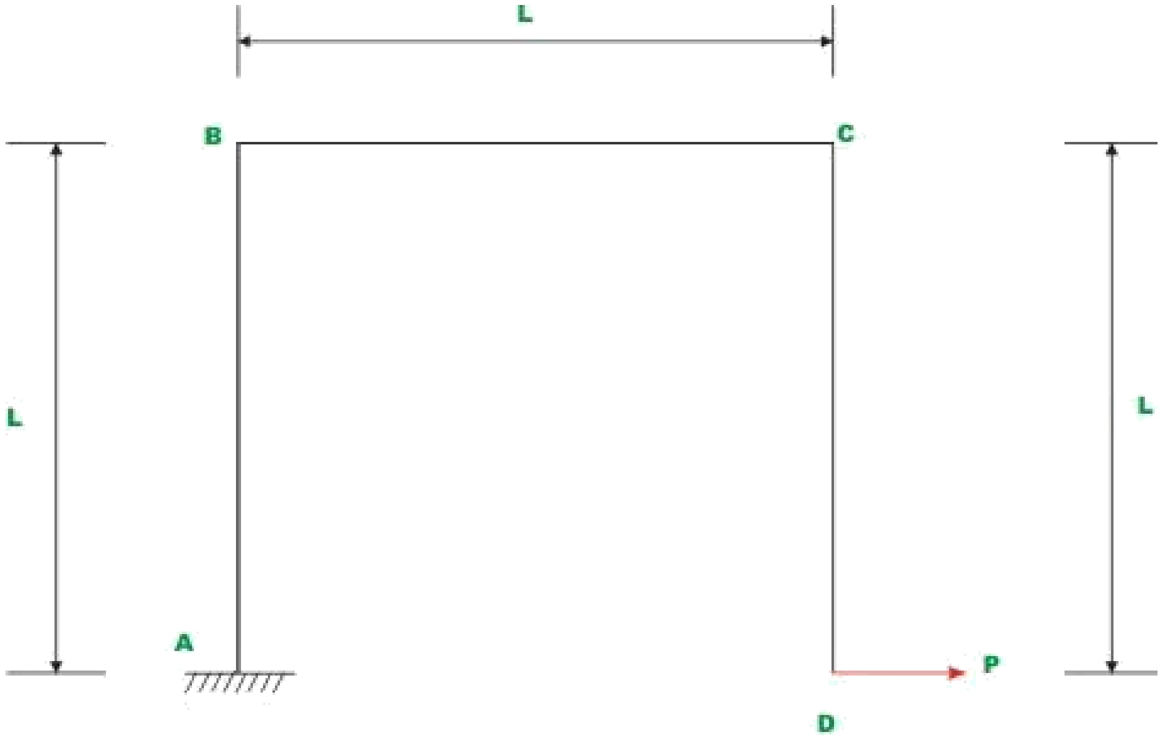


Fig. 3.4 Example 3.3

The deflection D may be obtained via. Castigliano's theorem. The beam segments BA and DC are subjected to bending moment Px ($0 < x < L$) and the beam element BC is subjected to a constant bending moment of magnitude PL .

Total strain energy stored in the frame due to bending

$$U = 2\int_0^L \frac{(Px)^2}{2EI} dx + \int_0^L \frac{(PL)^2}{2EI} dx \tag{1}$$

After simplifications,

$$U = \frac{P^2L^3}{3EI} + \frac{P^2L^3}{2EI} = \frac{5P^2L^3}{6EI} \tag{2}$$

Differentiating strain energy with respect to P we get,

$$\frac{\partial U}{\partial P} = u_D = 2\frac{5PL^3}{6EI} = \frac{5PL^3}{3EI}$$

Example 3.4

Find the vertical deflection at A of the structure shown Fig. 3.5. Assume the flexural rigidity EI and torsional rigidity GJ to be constant for the structure.

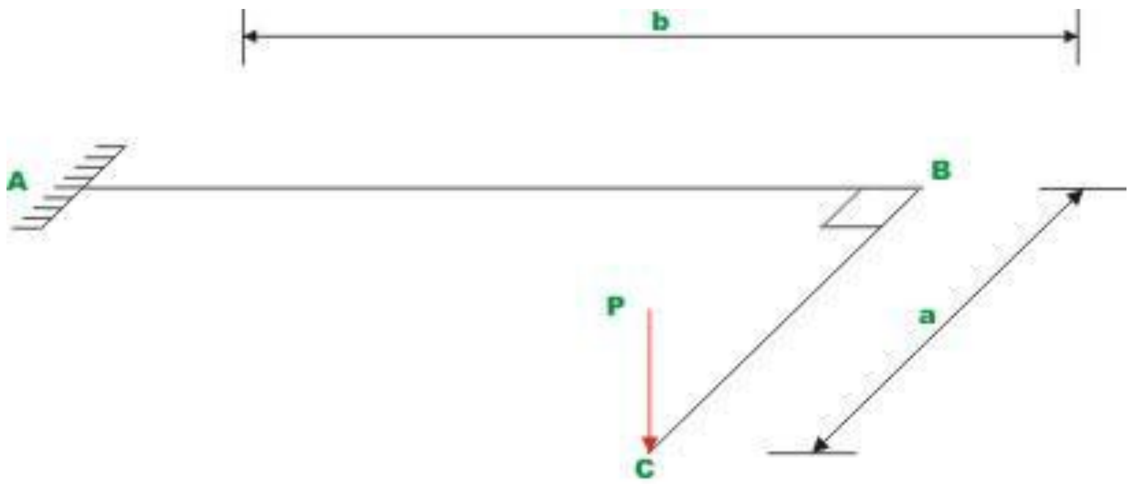


Fig.3.5 Example 3.4

The beam segment BC is subjected to bending moment Px ($0 < x < a$; x is measured from C)and the beam element AB is subjected to torsional moment of magnitude Pa and a bending moment of Px ($0 \le x \le b$; x is measured from B) . The strain energy stored in the beam ABC is,

$$U = \int_0^a \frac{M^2}{2EI}\,dx + \int_0^b \frac{(Pa)^2}{2GJ}\,dx + \int_0^b \frac{(Px)^2}{2EI}\,dx \tag{1}$$

After simplifications,

$$U = \frac{P^2 a^3}{6EI} + \frac{P^2 a^2 b}{2GJ} + \frac{P^2 b^3}{6EI} \tag{2}$$

Vertical deflection u_A at A is,

$$\frac{\partial U}{\partial P} = u_A = \frac{Pa^3}{3EI} + \frac{Pa^2 b}{GJ} + \frac{Pb^3}{3EI} \tag{3}$$

Example 3.5

Find vertical deflection at C of the beam shown in Fig. 3.6. Assume the flexural rigidity EI to be constant for the structure.

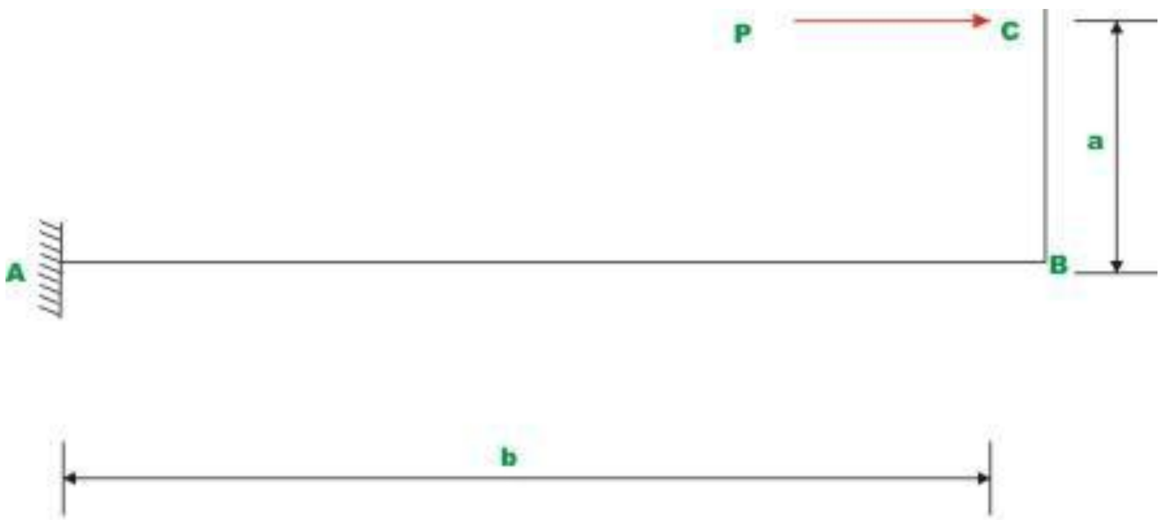


Fig. 3.6 Example 3.5

The beam segment CB is subjected to bending moment Px ($0 < x < a$) and beam element AB is subjected to moment of magnitude Pa .
To find the vertical deflection at C , introduce a imaginary vertical force Q at C .
Now, the strain energy stored in the structure is,

$$U = \int_0^a \frac{(Px)^2}{2EI} dx + \int_0^b \frac{(Pa + Qy)^2}{2EI} dy \qquad (1)$$

Differentiating strain energy with respect to Q , vertical deflection at C is obtained.

$$\frac{\partial U}{\partial Q} = u_C = \int_0^b \frac{2(Pa + Qy)\, y}{2EI} dy \qquad (2)$$

$$u_C = \frac{1}{EI} \int_0^b Pay + Qy^2 \, dy \qquad (3)$$

$$u_C = \frac{1}{EI}\left(\frac{Pab^2}{2} + \frac{Qb^3}{3}\right) \tag{4}$$

But the force Q is fictitious force and hence equal to zero. Hence, vertical deflection is,

$$u_C = \frac{Pab^2}{2EI} \tag{5}$$

Castiglione's Second Theorem

In any elastic structure having n independent displacements $u_1, u_2, ..., u_n$ corresponding to external forces $P_1, P_2, ..., P_n$ along their lines of action, if strain energy is expressed in terms of displacements then n equilibrium equations may be written as follows.

$$\frac{\partial U}{\partial u_j} = P_j, \quad j = 1, 2, ..., n \tag{3.9}$$

This may be proved as follows. The strain energy of an elastic body may be written as

$$U = \frac{1}{2} P_1 u_1 + \frac{1}{2} P_2 u_2 + ... + \frac{1}{2} P_n u_n \tag{3.10}$$

We know from Lesson 1 (equation 1.5) that

$$P_i = k_{i1} u_1 + k_{i2} u_2 + ... + k_{in} u_n, \quad i = 1, 2, .., n \tag{3.11}$$

where k_{ij} is the stiffness coefficient and is defined as the force at i due to unit displacement applied at j. Hence, strain energy may be written as,

$$U = \frac{1}{2} u_1 [k_{11} u_1 + k_{12} u_2 + ...] + \frac{1}{2} u_2 [k_{21} u_1 + k_{22} u_2 + ...] + ...+ \frac{1}{2} u_n [k_{n1} u_1 + k_{n2} u_2 + ...] \tag{3.12}$$

We know from reciprocal theorem $k_{ij} = k_{ji}$. Hence, equation (3.12) may be simplified as,

$$U = \frac{1}{2}\left(k_{11} u_1^2 + k_{22} u_2^2 + ... + k_{nn} u_n^2\right) + \left[k_{12} u_1 u_2 + k_{13} u_1 u_3 + ... + k_{1n} u_1 u_n\right] + ... \tag{3.13}$$

Now, differentiating the strain energy with respect to any displacement u_1 gives the applied force P_1 at that point, Hence,

$$\frac{\partial U}{\partial u_1} = k_{11} u_1 + k_{12} u_2 + \dots\dots\dots + k_{1n} u_n \tag{3.14}$$

Or,

$$\frac{\partial U}{\partial u_j} = P_j, \qquad j = 1, 2, \dots, n \tag{3.15}$$

Summary

In this lesson, Castiglione's first theorem has been stated and proved for linearly elastic structure with unyielding supports. The procedure to calculate deflections of a statically determinate structure at the point of application of load is illustrated with examples. Also, the procedure to calculate deflections in a statically determinate structure at a point where load is applied is illustrated with examples. The Castiglione's second theorem is stated for elastic structure and proved.