Isomorphic Graphs

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Isomorphism of Graphs

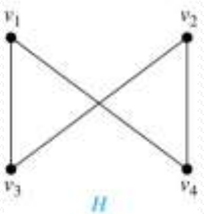
Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 , for all a and b in V_1 . Such a function f is called an *isomorphism.* Two simple graphs that are not isomorphic are called *non-isomorphic*.

Isomorphism of Graphs (*cont.*)

- Note
  - In short, out of the two isomorphic graphs, one is a tweaked version of the other. An unlabelled graph also can be thought of as an isomorphic graph.
  - If $G_1 \cong G_2$ then −
- $\rightarrow$ $|V(G_1)| = |V(G_2)|$
- $\rightarrow$ $|E(G_1)| = |E(G_2)|$
- Degree sequences of G_1 and G_2 are the same.
- If the vertices $\{V_1, V_2, ...V_k\}$ form a cycle of length k in G_1, then the vertices $\{f(V_1), f(V_2),\dots f(V_k)\}$ should form a cycle of length k in G_2.

Isomorphism of Graphs (*cont.*)

Example: Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.



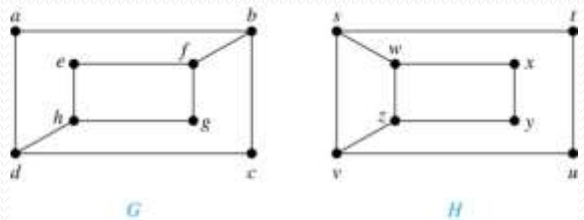
Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. Note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4. Each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in H.

Isomorphism of Graphs (*cont.*)

- It is difficult to determine whether two simple graphs are isomorphic using brute force because there are n! possible one-to-one correspondences between the vertex sets of two simple graphs with n vertices.
- The best algorithms for determining weather two graphs are isomorphic have exponential worst case complexity in terms of the number of vertices of the graphs.
- Sometimes it is not hard to show that two graphs are not isomorphic. We can do so by finding a property, preserved by isomorphism, that only one of the two graphs has. Such a property is called *graph invariant*.
- There are many different useful graph invariants that can be used to distinguish non-isomorphic graphs, such as the number of vertices, number of edges, and degree sequence (list of the degrees of the vertices in non increasing order).

Isomorphism of Graphs (*cont.*)

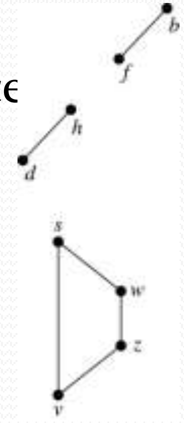
Example: Determine whether these two graphs are isomorphic.



Solution: Both graphs have eight vertices and ten edges.
They also both have four vertices of degree two and four of degree three.

However, *G* and *H* are not isomorphic. Note that since $deg(a) = 2$ in *G*, *a* must correspond to *t*, *u*, *x*, or *y* in H, because these are the vertices of degree 2. But each of these vertices is adjacent to another vertex of degree two in *H*, which is not true for *a* in *G*.

Alternatively, note that the subgraphs of *G* and *H* made up of vertice
degree three and the edges connecting them must be isomorphic.
But the subgraphs, as shown at the right, are not isomorphic.



Isomorphism of Graphs (*cont.*)

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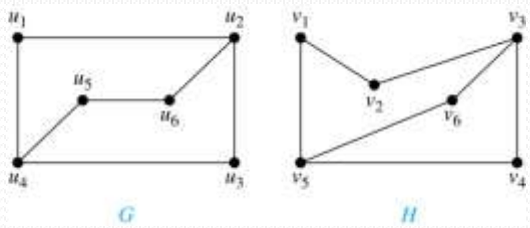


Solution: Both graphs have six vertices and seven edges.
They also both have four vertices of degree two and two of degree three.
The subgraphs of G and H consisting of all the vertices of degree two and the edges connecting them are isomorphic. So, it is reasonable to try to find an isomorphism f.

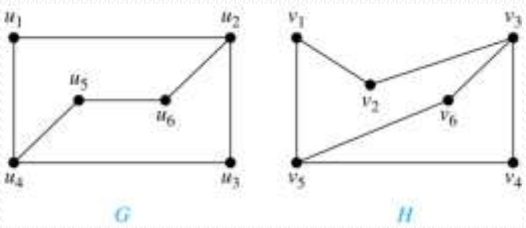
We define an injection f from the vertices of G to the vertices of H that preserves the degree of vertices. We will determine whether it is an isomorphism.

The function f with $f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$, and $f(u_4) = v_5$, $f(u_5) = v_1$, and $f(u_6) = v_2$ is a one-to-one correspondence between G and H. Showing that this correspondence preserves edges is straightforward, so we will omit the details here. Because f is an isomorphism, it follows that G and H are isomorphic graphs.

See the text for an illustration of how adjacency matrices can be used for this verification.

Isomorphism of Graphs (*cont.*)

Example: Determine whether these two graphs are isomorphic.



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Isomorphism of Graphs (*cont.*)

Theorem:

Two simple graphs G and H are isomorphic if and only if $\bar{G}$ is isomorphic to $\bar{H}$

Solution:

- If f is an isomorphism from G to H, then f is a vertex bijection preserving adjacency and non-adjacency, and hence f preserves non-adjacency and adjacency in $\bar{G}$ and is an isomorphism from $\bar{G}$ to $\bar{H}$. The same argument applies to the converse since the complement of $\bar{G}$ is G.

Note:

- All the above conditions are necessary for the graphs G_1 and G_2 to be isomorphic, but not sufficient to prove that the graphs are isomorphic.
  - ($G_1 \equiv G_2$) if and only if ($\bar{G}_1 \cong \bar{G}_2$) where G_1 and G_2 are simple graphs.
  - ($G_1 \equiv G_2$) if the adjacency matrices of G_1 and G_2 are the same.
  - ($G_1 \equiv G_2$) if and only if the corresponding subgraphs of G_1 and G_2(obtained by deleting some vertices in G_1 and their images in graph G_2) are isomorphic.

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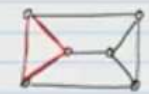
Isomorphism of Graphs (*conclusion*)

If two graphs are isomorphic then they have
the same number of vertices = 6
" " edges = 9
" " component = 1
the same degree sequence
diameter = 2
length of the longest path = 6

If two graphs differ in any of these respects, they are not isomorphic.
However, having all these values in common does not imply that two graphs are isomorphic.

 $\simeq$  

$G_1 = K_{3,3}$ G_2 G_3

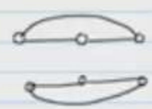
(3, 3, 3, 3, 3, 3) (3, 3, 3, 3, 3, 3)

degree sequence

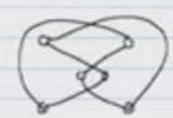
If two graphs are isomorphic and one of them contains a cycle of particular length, then the same must be true of the other graph.

$G_1 \not\simeq G_3$

(Th) G_1 & G_3 are isomorphic iff their complements are isomorphic.



$\bar{G}_1$



$\bar{G}_3$

$G_1 \simeq G_3$
$\bar{G}_1 \not\simeq \bar{G}_3$

$\bar{G}_1$ consists of two disjoint 3-cycles

$\bar{G}_3$ is a cycle of length 6.

Algorithms for Graph Isomorphism

- The best algorithms known for determining whether two graphs are isomorphic have exponential worst-case time complexity (in the number of vertices of the graphs).
- However, there are algorithms with linear average-case time complexity.
- You can use a public domain program called NAUTY to determine in less than a second whether two graphs with as many as 100 vertices are isomoprhic.
- Graph isomorphism is a problem of special interest because it is one of a few NP problems not known to be either tractable or NP-complete .

Applications of Graph Isomorphism

- The question whether graphs are isomorphic plays an important role in applications of graph theory. For example,
  - chemists use molecular graphs to model chemical compounds. Vertices represent atoms and edges represent chemical bonds. When a new compound is synthesized, a database of molecular graphs is checked to determine whether the graph representing the new compound is isomorphic to the graph of a compound that this already known.
  - Electronic circuits are modeled as graphs in which the vertices represent components and the edges represent connections between them. Graph isomorphism is the basis for
    - the verification that a particular layout of a circuit corresponds to the design's original schematics.
    - determining whether a chip from one vendor includes the intellectual property of another vendor.