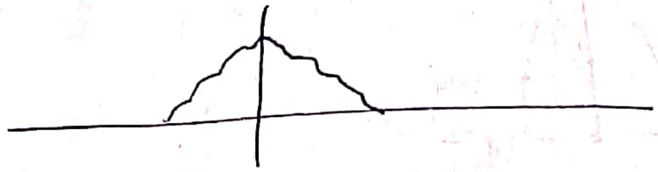
Fourier series applies only to periodic signals but a lot of times our signal is not periodic what about an aperiodic signal ?



Idea is to assume initially I've some copies of this signal at time T and then I take those copies to ∞ (i.e $T \to \infty$)

F.S rep.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

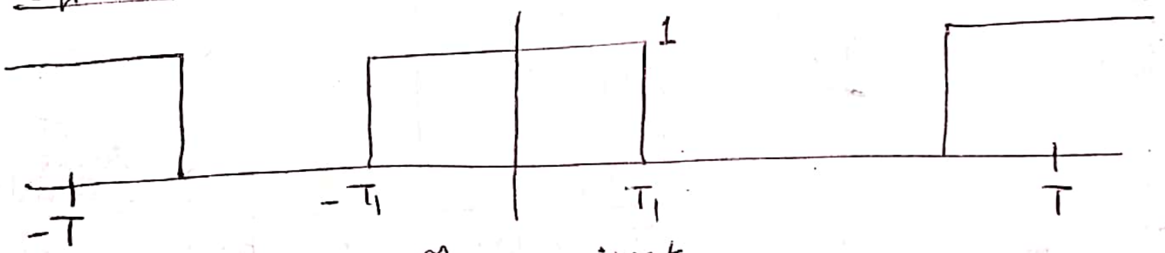
(bunch of complex sinusoids)

when T is very large then ω_0 is very small (i.e sinusoids (added together) are very close together in freq.)

$$* = \int_{\omega=-\infty}^{\infty} \underbrace{\boxed{}}_{F.T} e^{j\omega t} \, dt$$

instead of sampling the freq. axis finely, I've continous freq.

<u>Square wave :-</u>



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T_1}$$

(general form here, $T_1 \neq T/2$

$$T a_k = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0} = \left. \frac{2 \sin \omega T_1}{\omega} \right|_{\omega = k\omega_0}$$

what that means is that to get the a_k what I do is I take this continous funcn. of ω & I sample it every ω_0 unit

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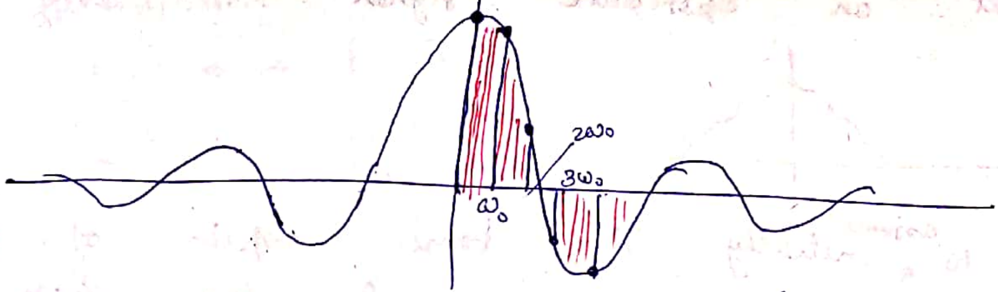


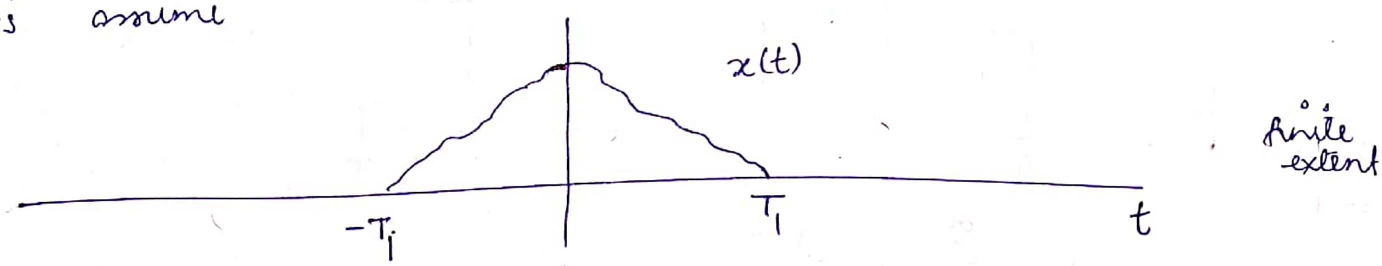
That's the nice way of thinking about how the Fourier series & F.T are related. It basically says that the underlying thing is the F.T & to get the Fourier series, I sample the F.T at these equally spaced values.

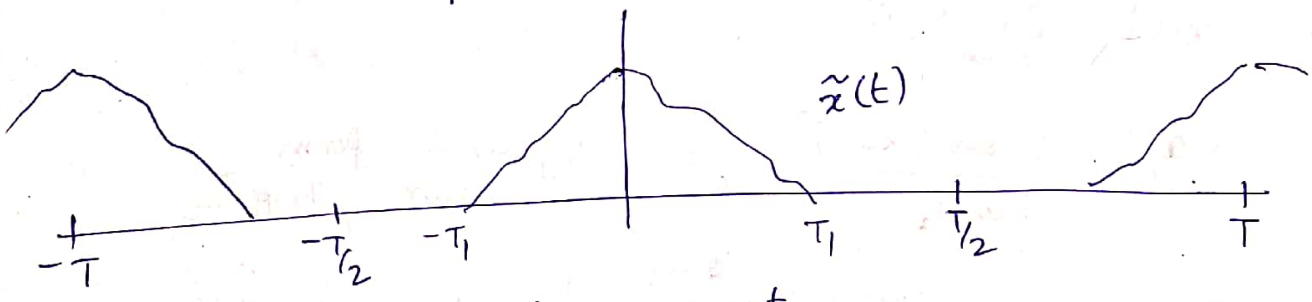
Intuition :-

$\{a_k\}$ are evenly spaced values of this continuous funcn. of ω

As $T \to \infty$, $\omega_0 \to 0$, samples get infinitely close (basically getting every pt. along this CT funcn.)

Let's assume





$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} \, dt$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt$$

for this integral I can replace $\tilde{x}$ by x because in this range $(-T/2$ to $T/2)$ these I are exactly same

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} X(\omega)\Big|_{\omega = k\omega_0} = \frac{1}{T} X(k\omega_0)$$

<u>Define</u>: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ (This continous funcn. evaluated at $\omega = k\omega_0$)

[The a_k's are related to this continous funcn. $X(\omega)$]

"The envelope" (Samples of this envelope) in previous picture

~~Con~~ Sinc → envelope func.

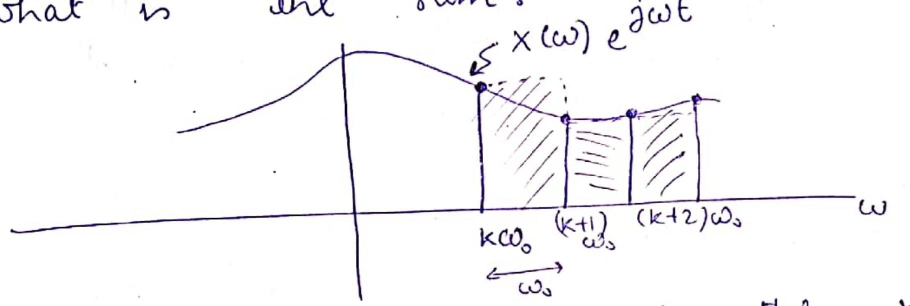
a_k → samples.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(k\omega_0)\, e^{jk\omega_0 t} \quad ; \omega_0 = \frac{2\pi}{T}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[X(k\omega_0) \cdot e^{jk\omega_0 t} \right] . \omega_0 \qquad \frac{1}{T} = \frac{\omega_0}{2\pi}$$

It's a complex funcn.

what is the sum?



sampling equally spaced multiples of ω_0

So, each of the term in the sum is the area of rect.X

(Width $= \omega_0$; height → the cont. funcn. sampled at $k\omega_0$)

integral → Riemann sum (make boxes narrower → reduce ω_0

$\because T \to \infty$, $\omega_0 \to$ ~~∞~~ 0

$\lim_{\omega \to 0}$ on both sides (just this aperiodic signal)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad \text{Inverse F.T}$$

<u>Recap</u>:

I defined $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ → Fourier Transf.

$x(t)$ (cont. funcn) → periodic → F.S (Discrete)

→ ~~no~~ aperiodic → F.T (cont. $X(\omega)$)

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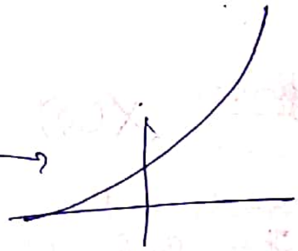
Discrete eq. of these 2

Q ~~dy~~ When does it work?

① $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ Finite energy



② Finite no. of extrema

③ Finite no. of discontinuities

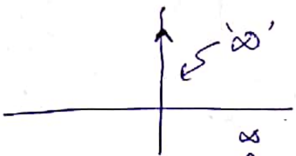
F.S is like a sp. case of F.T in some sense

(*) Exception; we do allow the F.T of periodic signals

eg:

It has a F.S but I can also allow to take F.T of it with the understanding that integral under the signal is going to be ∞.

1) $x(t) = \delta(t)$



'∞' height & unit area

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

→ The integral corresp. to where the delta funcn. is fired

$$= \int_{-\infty}^{\infty} e^{j\omega(0)} = 1 \quad \text{for all } \omega$$

F.T of δ funcn. is const.

2) $x(t) = \delta(t - t_0)$

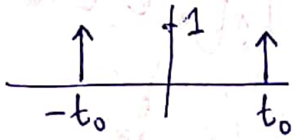
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0}$$

These funcn. have same mag. but ~~same~~ diff. phase.

The intuition just like F.S when I time shift the signal I phase shifted the F.S coeff. In same way I phase shift F.T. I'm not really changing the contribution of things. All one is doing is when these cosines & sines start.

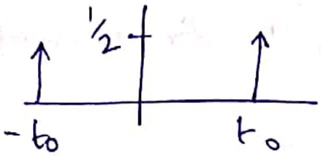
③



$\longleftrightarrow \quad e^{j\omega t_0} + e^{-j\omega t_0}$

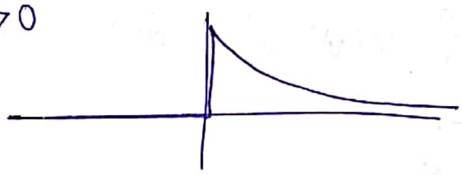
$= 2\cos(\omega t_0)$

Amplitude mod. (mult. by cos.)



$\rightarrow \cos \omega t_0$

④ $x(t) = e^{-at} u(t) \qquad a>0$



$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

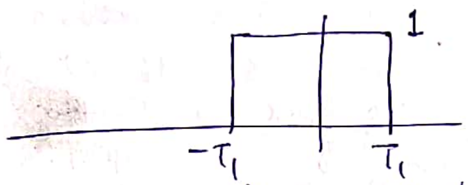
$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{-1}{a+j\omega} \cdot e^{-(a+j\omega)t} \Big|_{t=0}^{t=\infty}$$

$$= \frac{1}{a+j\omega}$$

(Lap. xform is more gen. than F.T)

5)



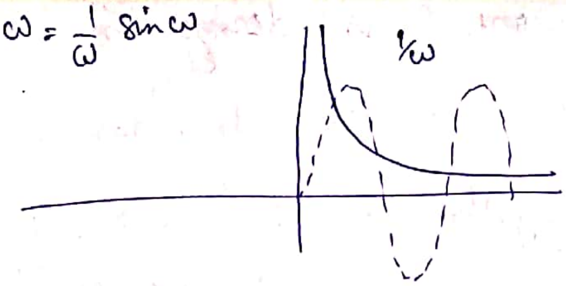
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

$$= \frac{-1}{j\omega} e^{-j\omega t}\Big|_{t=-T_1}^{t=T_1} = \frac{-1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right)$$

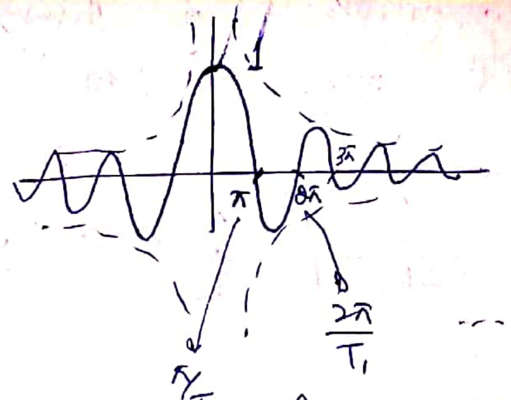
$$= \frac{2 \sin \omega T_1}{\omega} = 2T_1 \frac{\sin \omega T_1}{\omega T_1} = 2T_1 \operatorname{sinc} \omega T_1$$

pulse in time domain ⟹ sinc in freq dom. & vice versa

$\text{sinc}\,\omega = \frac{1}{\omega}\sin\omega$

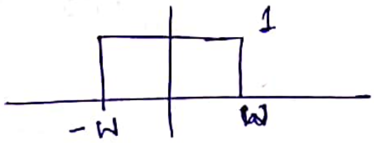


→



for $\text{sinc}\,\omega T_1$

what if $X(\omega) =$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi j t}\, e^{jt\omega}\Big]_{\omega=-W}^{\omega=W}$$

$$= \frac{1}{\pi t}\cdot\frac{1}{2j}\left(e^{jtW} - e^{-jtW}\right) = \frac{\sin tW}{\pi t}$$

$$= \frac{W}{\pi}\,\text{sinc}\,(tW)$$

Time Dom	Freq
Pulse	sinc
Sinc.	pulse

This is not coincidence.
This is what is called "Duality"

T	F
$\delta(t)$	1 (const)
const.	$\delta(\omega)$

Pair of impulse in T.D → cosine freq.

cosine in TD → pair imp ?

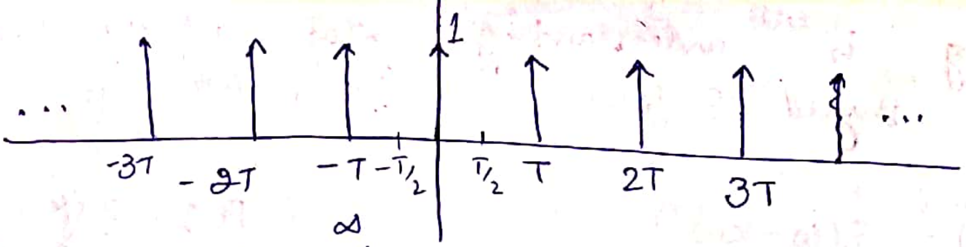
(2π nodding around, but shapes of the signals is what ~~matters~~ match!)

Pair of pulses in T.D ⟶ cosine in freq. dom.

cosine in T.D ⟶ pair of impulses in freq. dom

? → What if $x(t)$ is periodic?

We can take either the F.S or F.T (& intuition tells us that these should be very closely related)



(bunch of δ-funcn.)

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \rightarrow \text{impulse train}$$

now, I can do F.S to the signal (period T)

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t} \cdot dt$$

$$= \frac{1}{T} \quad \text{for all } k$$

in this region ($-T/2$ to $T/2$) δ-funcn. fires only at $t=0$ ∴ that's same as evaluating this at 0

impulse train gives const. F.S coeff.

F.T : $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t-kT)\, e^{j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty}$$

---

gen.

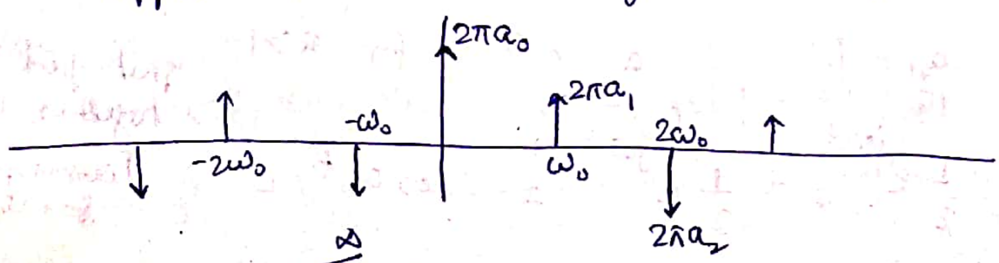
suppose $x(t)$ is a periodic signal so, it can be represented as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

F.T

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_0)$$

Let's suppose we're a signal whose F.T looks like



X(ω)

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0)$$

what is the corr. $x(t)$?

The key thing is understanding what is the inverse F.T of this shifted δ funcn.

Suppose, $X(\omega) = \delta(\omega - k\omega_0)$ — Freq. shift in freq. dom.

$$x(t) = \frac{1}{2\pi}\int \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} e^{jk\omega_0 t}$$

Phase ~~Freq~~ shift in time domain

So, for $X(\omega) = \sum 2\pi a_k \delta(\omega - k\omega_0)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$x(t) = \frac{1}{2\pi}\int \sum \delta(\omega - k\omega_0) . 2\pi a_k$

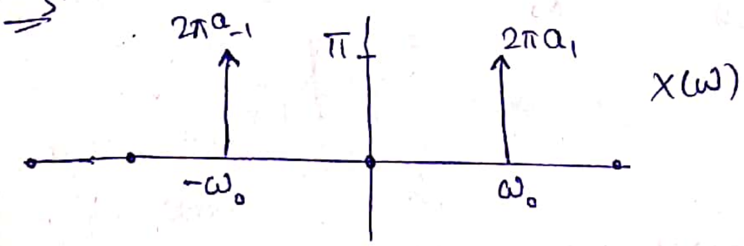
= Fourier series of $x(t)$ for a periodic signal with $T = \frac{2\pi}{\omega_0}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum 2\pi a_k \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega$$

$$= \sum \int_{-\infty}^{\infty} a_k \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega$$

$$= \sum a_k e^{jk\omega_0 t}$$

eg.



I can take it as

$\omega = 0 \Rightarrow X(\omega) = 0$

$\omega = \pm\omega_0 \Rightarrow X(\omega) =$ Delta of height π

$a_0 = 0$; $a_{\pm 1} = \frac{1}{2}$, $a_{\pm k} = 0$ for $k > 1$

$$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} = \cos \omega_0 t$$

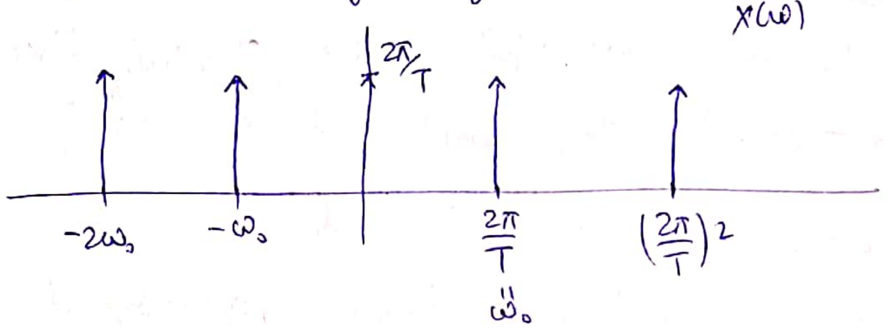
That just together into Fourier series

I can take the cosine ; I can compute the F.S of it & then I can put those F.S coeff. on top of these impulse & get the F.T

For a periodic impulse train Period T

F.S coeff $a_k = \frac{1}{T}$

So F.T of signal



impulse $2\pi \times a_k$ in height & spaced apart by ω_0

$\omega_0 = \frac{2\pi}{T}$

If u're impulse train in T.D ; what u get in freq. domain is also an Impulse train where impulses are sep. by $\frac{2\pi}{T}$ (The wider the impulses are in F.D , the closer they're together over here) in T.D)

That makes sense!

If I take T getting really large → not periodic at all → one impulse at the middle & nothing else will happen here that these The corr. thing will all push together so closely that they become a constant (∵ δ funcn in T.D → const. in F.D)

→ Fourier X-form properties

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\mathcal{F}(x(t)) = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$X($

$X(e^{j\omega}) \rightarrow X(j\omega)$ in Laplace xform $X(s)$

① Linearity

$x(t) \longleftrightarrow X(\omega)$ $y(t) \longleftrightarrow Y(\omega)$

$a\,x(t) + b\,y(t) \longleftrightarrow a\,X(\omega) + b\,Y(\omega)$

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② Time shift :-

$$x(t \pm t_0) \longleftrightarrow X(\omega)\, e^{\pm j\omega t_0}$$

$$|X(\omega)\, e^{-j\omega t_0}| = |X(\omega)|$$

mag. is same as before
· shifting the phase of sinusoids

Freq. shift:-

$$x(t) \cdot e^{\pm j\omega_0 t} \longleftrightarrow X(\omega \mp \omega_0)$$

③ Symmetry :-

$$X(\omega) = Re(\omega) + j\, Im(\omega)$$

If $x(t)$ is real,

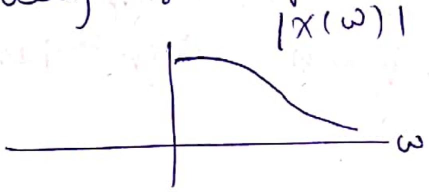
$Re(\omega) = Re(-\omega)$ Real part even

$Im(\omega) = -Im(-\omega)$ Img. part odd

$\Big]\; X(\omega) = [X(-\omega)]^*$

· $|X(\omega)|$ is even ; $\angle X(\omega)$ is odd

For that reason, we often deal with real signals
this tells us that I don't have to plot the
F.T for all values of ω ∵ mag. is even we
usually see plots that're like



stuff on left hand side because its mirror image of y axis

can show

$$Ev(x(t)) \longleftrightarrow Re(X(\omega))$$

$$Od(x(t)) \longleftrightarrow j\, Im(X(\omega))$$

$x(t)$ real, even , $X(\omega)$ is also real , even

$x(t)$ real, odd ; $X(\omega)$ is purely img. & odd

④ Differentiation / Integration

$$x'(t) \longleftrightarrow j\omega\, X(\omega)$$

[$sX(s)$ in L.T]

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

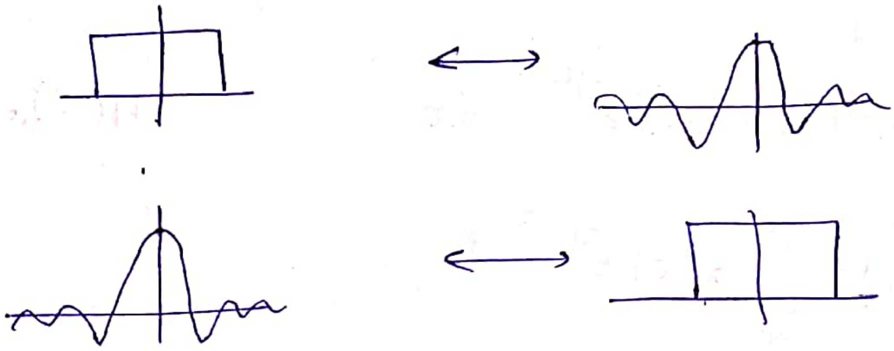
→ DC offset

Time scaling

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

shrink T.D signal spread F.D & vice versa

• Duality



$x(t) \longleftrightarrow X(\omega)$

$X(t) \longleftrightarrow 2\pi x(-\omega)$

Using duality principle find F.T of 1 ? & hence find F.T of $e^{j\omega_0 t}$

• Parseval

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Same total energy in either domain

Key principle is convolution :-

If $y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega) H(\omega)$

imp. resp.

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

eq. way of rep. is thru. Freq. resp.

$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$

Freq. resp. (Fourier xform of impulse resp.)

mul. things in Freq. domain & inverse F.T is lot easier than time dom. convolution