LECTURE # 04 - Topic 2 (Contd)

Recap & Review of Previous Lecture

→ Main purpose of Comm. System is to transmit message signals (baseband signals) from Tx over Communication channel to Rx with help of carrier

→ To understand this transmission procedure we should be able to understand the signals both in Time domain ~~(f(t))~~ & in frequency domain

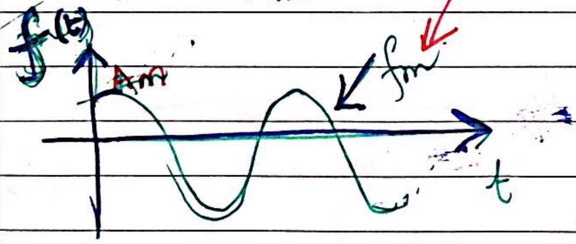
→ Time domain shows time variation of a signal

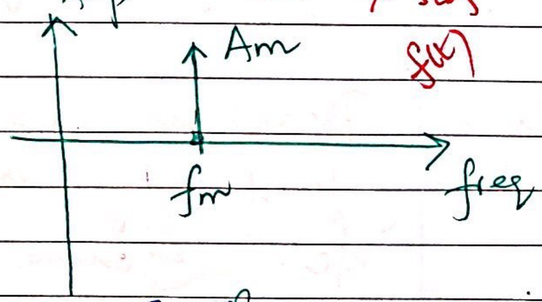
→ Frequency domain shows frequency spectrum of a signal

→ e.g

$f(t) = A_m \cos \omega_m t$ — Time domain representation of $f(t)$

$F(f)$ — Frequency domain representation of $f(t)$





$f(t)$ in Time domain $(f(t))$ | f In Frequency domain $(F(f))$

→ In general if $f(t)$ is time domain function then by applying its Fourier Xform we gets its frequency domain function

$$\therefore \quad f(t) \rightleftharpoons F(f)$$

Time domain function | Frequency domain function

Fourier Series Expansion

Let the function in time domain be $\rightarrow f(t)$
Then the function in frequency domain is $\rightarrow F(f)$

where $f(t) \rightleftharpoons F(f)$

or

$$F(f) = \mathcal{F}[f(t)]$$

- Frequency domain function.
- $F(f)$ is a function of frequency

[Fourier Transform operation]

- Time domain function
- $f(t)$ is a function of time

or may also be written as $F(\omega)$ where $\boxed{\omega = 2\pi f}$

2.1 What is Fourier Series ?

A function $f(t)$ can be represented by Fourier series which is an infinite series of function, each function related to the fundamental

We say that these functions are harmonically related to the fundamental.

We may write the fourier series expansion of $f(t)$ as Eq (2.1.a) below :-

Fourier Series expansion of $f(t)$

$$f(t) = A_0 + A_1 \cos(\omega_0)t + A_2 \cos(2\omega_0)t + \cdots + A_n \cos(n\omega_0)t + \cdots + B_1 \sin(\omega_0)t + B_2 \sin(2\omega_0)t + \cdots + B_n \sin(n\omega_0)t \ldots \quad (2.1.a)$$

Here $\omega_0 = 2\pi f_0 \rightarrow$ fundamental frequency

Angular frequency

$\omega_0 = \frac{2\pi}{T_0}$ where $f_0 = \frac{1}{T_0}$

T_0: Basic time Period of time domain function

Camlin

Scanned with CamScanner

Also A_0, A_n & B_n are the coefficients which have to be determined.

Example Let $f(t)$ → Square Waveform · periodic function

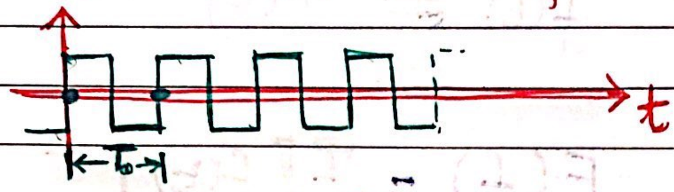


Fig 2.1

The above illustration shows a time domain function $f(t)$. It is a periodic square function. Thus we note from above:

- → $f(t)$ above is periodic function in time domain
- → $f(t)$ is a Square function waveform
- → $f(t)$ above is odd function ∵ $f(-t) = -f(t)$
- → $f(t)$ occurs with periodicity $T_0 = \frac{1}{f_0}$

ie $\omega_0 = 2\pi f_0$

or $\omega_0 = \frac{2\pi}{T_0}$

By applying (Fourier's Series rules) to the above function the square wave can also be written as Equation (2.1.a) below :-

$$f(t) = A_0 + A_1 \cos(\omega_0)t + A_2 \cos(2\omega_0)t + \dots + A_n \cos(n\omega_0)t + \dots + B_1 \sin(\omega_0)t + B_2 \sin(2\omega_0)t + \dots + B_n \sin(n\omega_0)t \quad (2.1.a)$$

or

as Equation (2.1.b)

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0)t + \sum_{n=1}^{\infty} B_n \sin(n\omega_0)t \quad (2.1.b)$$

& A_0, A_n, B_n are the coefficients to be determined.

Scanned with CamScanner

Example Let f(t) = Non-periodic Square function or any other non-periodic pulse (See Fig 2.2.(a,b,c))

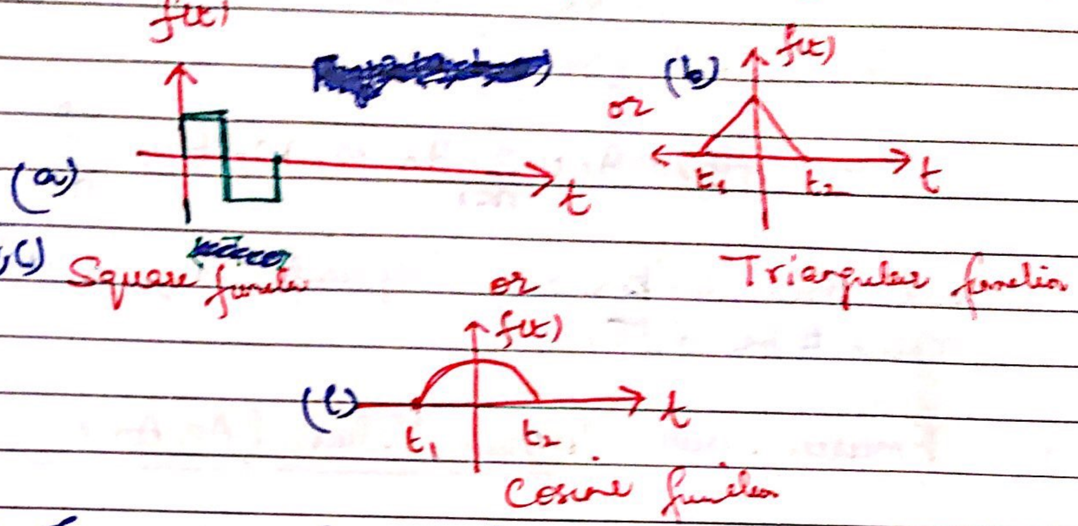


Fig 2.2(a,b,c)

→ The above illustration show non-periodic functions ie a square function, a triangular function & a cosine function respectively.

→ These are some examples of non-periodic (also called aperiodic) functions.

→ To determine such non-periodic functions in frequency domain we apply rules of Fourier Transform ($\mathcal{F}$) so that

$$f(t) \rightleftharpoons F(f)$$

are called the fourier transform pair.

Thumbs rule ⤵
#2.1 To sum it all up.

f(t)	Rules to be applied	F(ω) Frequency domain Spectrum
→ Periodic	Fourier Series Operation	Discrete spectrum
→ Non-Periodic	Fourier Transform Operation	Continuous Spectrum

(PTO)

Scanned with CamScanner

First we shall take up :—

2.2 Periodic Functions & Fourier Series Operation

Let $f(t)$ be a periodic function of periodicity $T_0 = \frac{1}{f_0}$ where $f(t)$ can be expressed as Eq. (2.1.a) (pg 24) as Eq. (2.1.b) (See pg 24)

ie
$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0)t + \sum_{n=1}^{\infty} \sin(n\omega_0)t \quad \ldots (2.1.b)$$

Here the unknown Coeffcients A_0, A_n & B_n have to be determined

~~2.2.1~~ Fourier Series Coeffcient Values (A_0, A_n, B_n)

2.2.1 General formulae (Remember $T_0 = \frac{1}{f_0}$ is Time period of $f(t)$)

$$A_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t)\,dt \quad \ldots (2.2.a)$$

$$A_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(t) \cos(n\omega_0)t\,dt \quad \ldots (2.2.b)$$

$$B_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(t) \sin(n\omega_0)t\,dt \quad \ldots (2.2.c)$$

2.2.2 Waveform Symmetries as related to Fourier series Coeffcients

ie. If $f(t)$ is **Even function** $[f(-t) = f(t)]$.
Then

$$A_0 = \frac{2}{T_0} \int_0^{T_0/2} f(t)\,dt \quad \ldots (2.3.a)$$

$$A_n = \frac{4}{T_0} \int_0^{T_0/2} f(t) \cos(n\omega_0)\,dt \quad \ldots (2.3.b)$$

$$B_n = 0 \quad \ldots (2.3.c)$$

Scanned with CamScanner

2.2.3 **Waveform Symmetries as related to Fourier Series Coefficients ie**

If $f(t)$ is an odd function $[f(-t) = -f(t)]$

Then

$$A_0 = 0 \qquad \ldots (2.4.a)$$

$$A_n = 0 \qquad \ldots (2.4.b)$$

$$B_n = \frac{4}{T_0}\int_0^{T_0/2} f(t) \sin(n\omega_0)t \, dt \qquad \ldots (2.4.c)$$

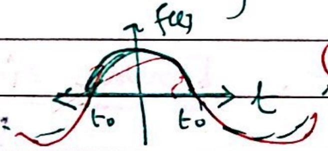
Thumbs rule # 2.2: Fourier Series expansion of an Even periodic function contains only cosine terms & a constant ($\because B_n = 0$)

Thumbs rule # 2.3: Fourier series expansion of an odd periodic function contains only sine terms ($\because A_0 = 0, A_n = 0$)

Thumbs rule # 2.4 $f(t)$ is an even function

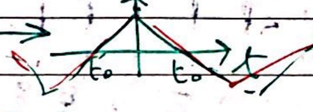
if $f(-t) = f(t)$

e.g $f(t) =$ Cosine fn



$f(t) =$ Square fn →

$f(t) =$ Triangular fn →



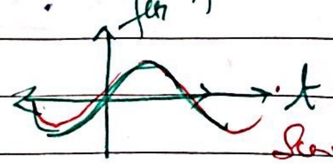
$f(t) = t^2$

etc

Thumbs rule # 2.5 $f(t)$ is an odd function

if $f(-t) = -f(t)$

e.g $f(t) =$ Sin function



$f(t) = t^3$

Note that $\cos(-\theta) = \cos\theta$ → Even fn

$\sin(-\theta) = -\sin(\theta)$ → odd fn

Camlin

Scanned with CamScanner

2.2.4 Frequency Domain Spectrum of f(t).

After having determined the Fourier Series Coefficients A_0, A_n & B_n by using the previously mentioned relevant formulae (Eq. (2.2 a,b,c); (2.3, a,b,c) & (2.4, a,b,c)) we can write the fourier series expansion of f(t).

This Fourier Series of a particular function (periodic) f(t) helps us in further getting the Frequency domain information of f(t).

The Frequency Domain Spectrum can be :-

2.2.4(a)
- Single Sided Amplitude Spectrum
- Single Sided Phase Spectrum

2.2.4(b)
- Double Sided Amplitude Spectrum
- Double Sided Phase Spectrum

Following illustrations will help in giving a brief idea of the Spectrums plotted in frequency domain.

These Concepts shall become more clear when we actually take up some typical Numericals to find f(t) series related A_0, A_n & B_n and subsequently then determine the Single Sided Amplitude and phase spectrum & Double Sided Amplitude and phase Spectrum.

P.T.O for the illustrations

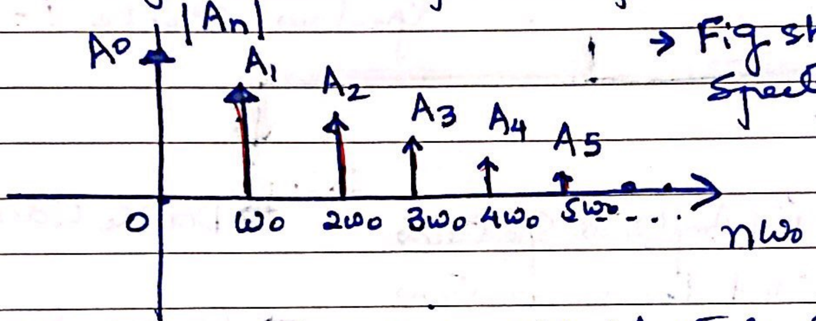
Scanned with CamScanner

→ Let $f(t)$ be a periodic function.
→ Assume the A_0, A_n & B_n has been determined
→ Writing the Fourier series expansion of $f(t)$,

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0)t + \sum_{n=1}^{\infty} B_n(n \sin \omega_0)t \quad \cdots (2.1.b)$$

$$\omega_0 = \frac{2\pi}{T_0}.$$

Single Sided Amplitude Spectrum [Fig 2.3.a].



→ Fig shows Equispaced Spectral Components

Fig 2.3.a : The Single Sided Amplitude Spectrum has:
→ Equispaced Spectral Components
→ Spacing is $\omega_0 = \frac{2\pi}{T_0}$ where T_0 is periodicity of $f(t)$.
→ At $\omega_0 = 0$, amplitude A_0 is the dc component.
→ All components are oscillating Cosines but A_0 is stationary ∴ it is a dc component.

Single Sided phase Spectrum [Fig 2.3.b]

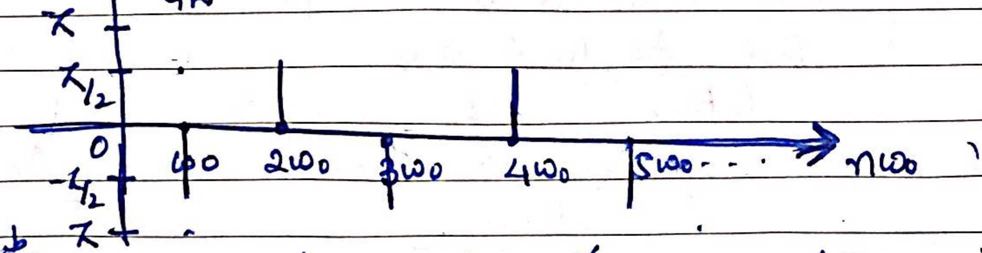


Fig 2.3.b
- Equispaced phase Components gives phase Information with reference to Cosine

~~Tutorial Sheet #02 (Contd.) Topic 2~~

~~Q-8~~

Double sided Amplitude spectrum [Fig 2.4.a]

A Double sided Spectrum is generated after applying Euler's theorem to the f(t) Series resulting In modified form of f(t).

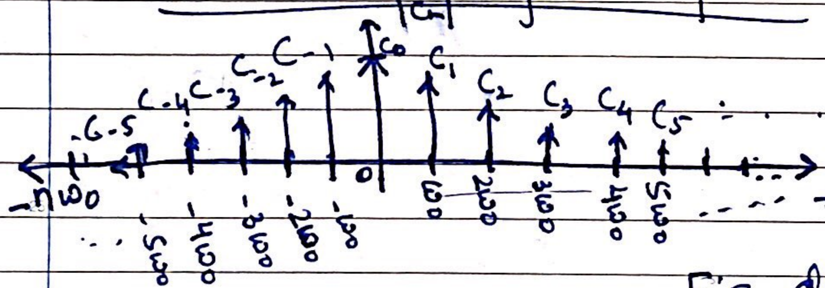
$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \exp[jn\omega_0]t + \sum_{n=1}^{\infty} C_{-n} \exp[-jn\omega_0]t \quad \text{--- (2-4)}$$

where $C_0 = A_0$ is the dc component as determined by (2.2.a).

$$C_n = \frac{A_n - jB_n}{2} \quad \& \quad C_{-n} = \frac{A_n + jB_n}{2}$$

$$|C_n| = \frac{\sqrt{A_n^2 + B_n^2}}{2}$$ where A_n, B_n is as defined earlier

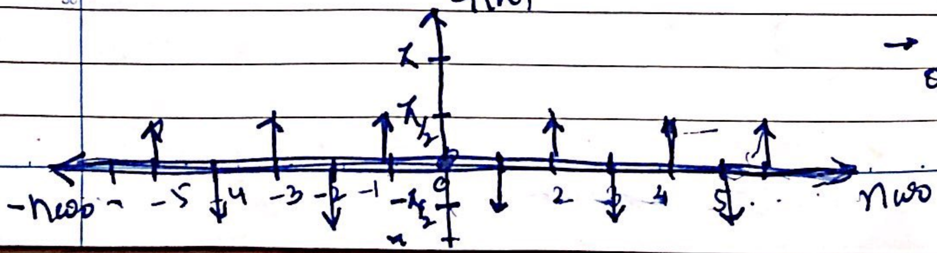
∴ Double sided Amplitude Spectrum [Fig 2.4.a]



$|C_n| = |C_{-n}|$

Fig displays Even Symmetry

Double sided phase spectrum - [Fig 2.4.b]



→ Fig displays odd symmetry.

→ $n\omega_0$ is also Marked as simple no

Thumb rule # 2.6.

In the single sided amplitude spectrum each component at $n\omega_0$ ($n = 1, 2, 3 \ldots$) represent oscillating cosines. e.g a component A_2 at $2\omega_0$ is oscillating with frequency $2\omega_0$. and a component ~~of~~ A_5 at $5\omega_0$ is oscillating with angular frequency $5\omega_0$. This implies that A_5 at $5\omega_0$ is oscillating faster than A_2 at $2\omega_0$. The spectrum is discrete

Thumb rule # 2.7

We regard amplitude as always being a positive quantity (ie $|A_n|$). When -ve sign appears it must be absorbed in phase spectrum

Thumb rule # 2.8

Phase angle of each component at $n\omega_0$ ($n = 1, 2, \ldots$) shall be measured w.r.t cosine.

Thumb rule # 2.9

In double sided spectrum (amplitude) each component at $n\omega_0$ & $-n\omega_0$ ($n = 1, 2, 3 \ldots$) represent rotating phasors. Hence a component eg at $3\omega_0$ and $-3\omega_0$ is rotating faster than the component at $+2\omega_0$ & $-2\omega_0$. The double sided spectrum exist for both +ve and -ve frequencies. This spectrum is also discrete and it exhibits even symmetry.

Thumb rule # 2.10

The double sided phase spectrum exhibits odd symmetry e.g if phase of $3\omega_0$ is $-\pi/2$ then phase of component at $-3\omega_0$ is $+\pi/2$

Thumb rule # 2.11

Amplitude of $|C_n| = \frac{1}{2}|A_n|$

Camlin

Scanned with CamScanner

So ~~summarize~~ brief it all.

→ $\cos(n\omega_0 t)$ is a cosine oscillating with freq. $n\omega_0$

→ $\exp(jn\omega_0 t)$ is a rotating phasor and it is rotating counter clockwise with angular freq. $n.\omega_0$

→ $\exp(-jn\omega_0 t)$ is a rotating phasor rotating clockwise with angular freq. $n\omega_0$.

→ Single sided spectrum represent oscillating cosines. Double sided spectrum represent rotating phasors

→ In a single sided spectrum the infinite summation of the oscillating cosines at speeds which are multiples of ω_0, sum up to give original f(t).

→ In double sided spectrum the infinite summation of the rotating phasors (at $+n\omega_0$ & $-n\omega_0$) which are multiples of ω_0, sum up to give original f(t).

Some Handy Algebraic formulae / expression that shall be used to get Spectrum of f(t).

Euler's theorem: $\exp(j\theta) = \cos(\theta) + j\sin(\theta)$

Eqn (2.5)

$$\left\{\begin{aligned} \exp(-j\theta) &= \cos(\theta) - j\sin(\theta) \\ \cos(\theta) &= \frac{\exp(j\theta) + \exp(-j\theta)}{2} \\ \sin(\theta) &= \frac{\exp(j\theta) - \exp(-j\theta)}{2j} \end{aligned}\right.$$

HENCE →

Eqn (2.6)

$$\cos(n\omega_0)t = \frac{\exp(jn\omega_0)t + \exp(-jn\omega_0)t}{2}$$

$$\sin(n\omega_0)t = \frac{\exp(jn\omega_0)t - \exp(-jn\omega_0)t}{2j}$$

Eqn 2.7

$$\exp(+j\pi) = \cos\pi + j\cancel{\sin\pi} = -1$$

$$\exp(-j\pi) = \cos\pi - j\cancel{\sin\pi} = -1$$

This implies $\boxed{\exp(\pm j\pi) = -1}$

Eqn 2.8.

$$\exp(+j\pi/2) = \cancel{\cos\pi/2} + j\sin\pi/2 = +j$$

$$\exp(-j\pi/2) = \cos\pi/2 - j\sin(-\pi/2) = -j$$

This implies $\boxed{\begin{array}{l}\exp(+j\pi/2) = +j \\ \exp(-j\pi/2) = -j\end{array}}$

P.T.O for T.Sheet #02 (Contd)

Tutorial Sheet # 02 (Contd.) Topic #2

Q.8 Let $f(t) = A_0 \cos \omega_0 t$

(a) Draw the above function in time domain
(b) Draw its Amplitude Spectrum → Single sided
(c) Draw its phase spectrum → Single sided

Q.9. Let $f(t) = A_0 \sin \omega_0 t$

(a) Draw the above function in time domain
(b) Draw its Amplitude Spectrum → Single sided
(c) Draw its phase spectrum → Single sided

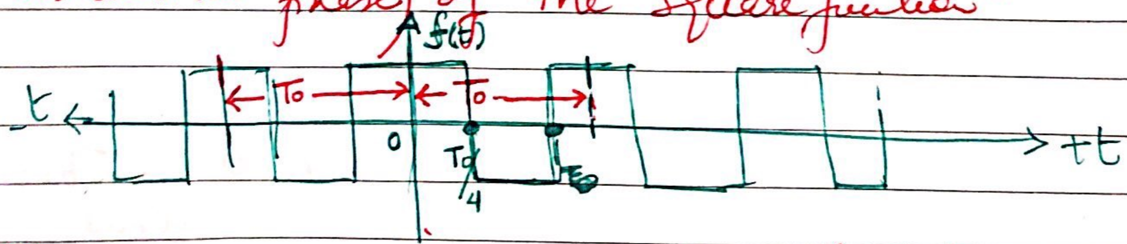
Q.10 Let $f(t) = -A_0 \cos \omega_0 t$

(a) Draw above function in time domain
(b) Draw its amplitude spectrum → Single sided
(c) Draw its phase spectrum → Single sided

Q.11 Let $f(t) = -A_0 \sin \omega_0 t$

(a) Draw above function in time domain.
(b) Draw its Amplitude Spectrum → Single sided
(c) Draw its phase spectrum → Single sided

Q.12 Plot the single sided discrete spectrum (Amplitude as well as phase) of the square function



Q.13 Plot the double sided discrete spectrum (Amplitude as well as phase) of the above square function