CURVED BEAMS

The theory of beam bending, presented in Chapter 7, is limited to straight beams or to beams that are mildly curved relative to their depth. However, if the ratio of the radius of curvature to depth for a beam is less than 5, the flexure formula (Eq. 7.1) is generally inadequate for describing the flexural stresses in the beam. For beams that are curved in such a manner, the theory of bending must also include consideration of the curvature. Such a theory is developed in this chapter based on mechanics of materials methods. Two important differences with respect to straight-beam bending result. First, the flexural stress distribution in a curved beam is nonlinear. Based on this result the neutral axis will not coincide with the centroidal axis of the cross section when the beam is subjected to pure bending. Second, a curved beam carries radial stresses as a consequence of the internal bending moment. These radial stresses have important design implications for thin-wall cross sections and for materials (such as wood and unidirectional composites) with relatively low tensile strength in the radial direction.

9.1 INTRODUCTION

Timoshenko and Goodier (1970) presented a solution based on the theory of elasticity for the linear elastic behavior of curved beams of rectangular cross sections for the loading shown in Figure 9.1*a*. They obtained relations for the radial stress σ_{rr}, the circumferential stress $\sigma_{\theta\theta}$, and the shear stress $\sigma_{r\theta}$ (Figure 9.1*b*). However, most curved beams do not

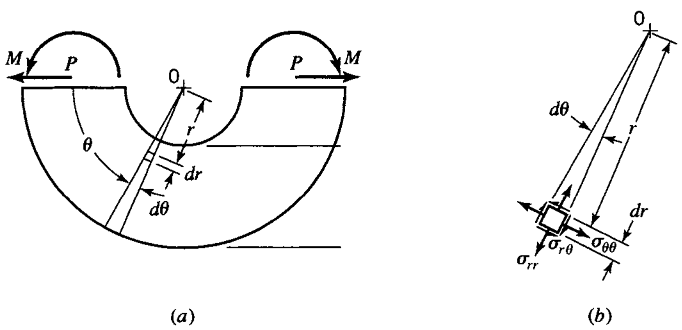


FIGURE 9.1 Rectangular section curved beam. (*a*) Curved beam loading. (*b*) Stress components.

have rectangular cross sections. Therefore, in Section 9.2 we present an approximate curved beam solution that is generally applicable to all symmetrical cross sections. This solution is based on two simplifying assumptions: 1. plane sections before loading remain plane after loading and 2. the radial stress σ_{rr} and shear stress $\sigma_{r\theta}$ are sufficiently small so that the state of stress is essentially one dimensional. The resulting formula for the circumferential stress $\sigma_{\theta\theta}$ is the curved beam formula.

9.2 CIRCUMFERENTIAL STRESSES IN A CURVED BEAM

Consider the curved beam shown in Figure 9.2*a*. The cross section of the beam has a plane of symmetry and the polar coordinates (r, θ) lie in the plane of symmetry, with origin at 0, the center of curvature of the beam. We assume that the applied loads lie in the plane of symmetry. A positive moment is defined as one that causes the radius of curvature at each section of the beam to increase in magnitude. Thus, the applied loads on the curved beams in Figures 9.1 and 9.2*a* cause positive moments. We wish to determine an approximate formula for the circumferential stress distribution $\sigma_{\theta\theta}$ on section *BC*. A free-body diagram of an element *FBCH* of the beam is shown in Figure 9.2*b*. The normal traction *N*, at the centroid of the cross section, the shear *V*, and moment M_x acting on face *FH* are shown in their positive directions. These forces must be balanced by the resultants due to the normal

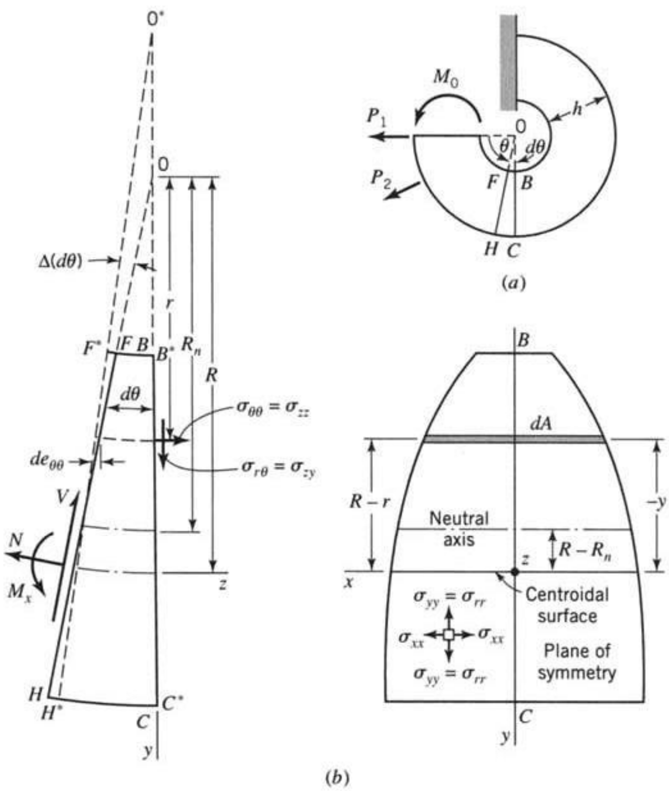


FIGURE 9.2 Curved beam.

stress $\sigma_{\theta\theta}$ and shear stress $\sigma_{r\theta}$ that act on face BC. The effect of the shear stress $\sigma_{r\theta}$ on the computation of $\sigma_{\theta\theta}$ is usually small, except for curved beams with very thin webs. However, ordinarily, practical curved beams are not designed with thin webs because of the possibility of failure by excessive radial stresses (see Section 9.3). Therefore, neglecting the effect of $\sigma_{r\theta}$ on the computation of $\sigma_{\theta\theta}$ is reasonable.

Let the z axis be normal to face BC (Figure 9.2b). By equilibrium of forces in the z direction and of moments about the centroidal x axis, we find

$$\sum F_z = \int \sigma_{\theta\theta}\, dA - N = 0$$

$$\sum M_x = \int \sigma_{\theta\theta}(R-r)\, dA - M_x = 0$$

or

$$N = \int \sigma_{\theta\theta}\, dA \tag{9.1}$$

$$M_x = \int \sigma_{\theta\theta}(R-r)\, dA \tag{9.2}$$

where R is the distance from the center of curvature of the curved beam to the centroid of the beam cross section and r locates the element of area dA from the center of curvature. The integrals of Eqs. 9.1 and 9.2 cannot be evaluated until $\sigma_{\theta\theta}$ is expressed in terms of r. The functional relationship between $\sigma_{\theta\theta}$ and r is obtained from the assumed geometry of deformation and stress–strain relations for the material.

The curved beam element $FBCH$ in Figure 9.2b represents the element in the undeformed state. The element $F^*B^*C^*H^*$ represents the element after it is deformed by the loads. For convenience, we have positioned the deformed element so that face B^*C^* coincides with face BC. As in the case of straight beams, we assume that planes B^*C^* and F^*H^* remain plane under the deformation. Face F^*H^* of the deformed curved beam element forms an angle $\Delta(d\theta)$ with respect to FH. Line F^*H^* intersects line FH at the neutral axis of the cross section (axis for which $\sigma_{\theta\theta} = 0$) at distance R_n from the center of curvature. The movement of the center of curvature from point 0 to point 0* is exaggerated in Figure 9.2b to illustrate the geometry changes. For infinitesimally small displacements, the movement of the center of curvature is infinitesimal. The elongation $de_{\theta\theta}$ of a typical element in the θ direction is equal to the distance between faces FH and F^*H^* and varies linearly with the distance $(R_n - r)$. However, the corresponding strain $\epsilon_{\theta\theta}$ is a nonlinear function of r, since the element length $r\, d\theta$ also varies with r. This fact distinguishes a curved beam from a straight beam. Thus, by Figure 9.2b, we obtain for the strain

$$\epsilon_{\theta\theta} = \frac{de_{\theta\theta}}{r\, d\theta} = \frac{(R_n - r)\Delta(d\theta)}{r\, d\theta} = \left(\frac{R_n}{r} - 1\right)\omega \tag{9.3}$$

where

$$\omega = \frac{\Delta(d\theta)}{d\theta} \tag{9.4}$$

It is assumed that the transverse normal stress σ_{xx} is sufficiently small so that it may be neglected. Hence, the curved beam is considered to be a problem in plane stress. Although radial stress σ_{rr} may, in certain cases, be of importance (see Section 9.3), here we neglect its effect on $\epsilon_{\theta\theta}$. Then, by Hooke's law, we find

$$\sigma_{\theta\theta} = E\epsilon_{\theta\theta} = \frac{R_n - r}{r}E\omega = \frac{E\omega R_n}{r} - E\omega \tag{9.5}$$

Substituting Eq. 9.5 into Eqs. 9.1 and 9.2, we obtain

$$N = R_n E\omega \int \frac{dA}{r} - E\omega \int dA = R_n E\omega A_m - E\omega A \tag{9.6}$$

$$\begin{aligned} M_x &= R_n R E\omega \int \frac{dA}{r} - (R + R_n)E\omega \int dA + E\omega \int r dA \\ &= R_n R E\omega A_m - (R + R_n)E\omega A + E\omega R A = R_n E\omega (R A_m - A) \end{aligned} \tag{9.7}$$

where A is the cross-sectional area of the curved beam and A_m has the dimensions of length and is defined by the relation

$$A_m = \int \frac{dA}{r} \tag{9.8}$$

Equation 9.7 can be rewritten in the form

$$R_n E\omega = \frac{M_x}{R A_m - A} \tag{9.9}$$

Then substitution into Eq. 9.6 gives

$$E\omega = \frac{A_m M_x}{A(R A_m - A)} - \frac{N}{A} \tag{9.10}$$

The circumferential stress distribution for the curved beam is obtained by substituting Eqs. 9.9 and 9.10 into Eq. 9.5 to obtain the curved beam formula

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M_x(A - rA_m)}{Ar(RA_m - A)} \tag{9.11}$$

The normal stress distribution given by Eq. 9.11 is hyperbolic in form; that is, it varies as $1/r$. For the case of a curved beam with rectangular cross section ($R/h = 0.75$) subjected to pure bending, the normal stress distribution is shown in Figure 9.3.

Since Eq. 9.11 has been based on several simplifying assumptions, it is essential that its validity be verified. Results predicted by the curved beam formula can be compared with those obtained from the elasticity solution for curved beams with rectangular sections and with those obtained from experiments on, or finite element analysis of, curved beams with other kinds of cross sections. The maximum value of circumferential stress $\sigma_{\theta\theta(\mathrm{CB})}$

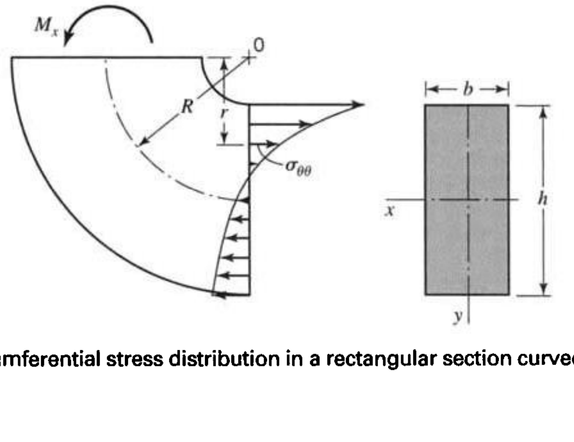


FIGURE 9.3 Circumferential stress distribution in a rectangular section curved beam ($R/h = 0.75$).

as given by the curved beam formula may be computed from Eq. 9.11 for curved beams of rectangular cross sections subjected to pure bending and shear (Figure 9.4). For rectangular cross sections, the ratios of $\sigma_{\theta\theta(\mathrm{CB})}$ to the elasticity solution $\sigma_{\theta\theta(\mathrm{elast})}$ are listed in Table 9.1 for pure bending (Figure 9.4*a*) and for shear loading (Figure 9.4*b*), for several values of the ratio R/h, where h denotes the beam depth (Figure 9.2*a*). The nearer this ratio is to 1, the less error in Eq. 9.11.

The curved beam formula is more accurate for pure bending than shear loading. The value of R/h is usually greater than 1.0 for curved beams, so that the error in the curved beam formula is not particularly significant. However, possible errors occur in the curved beam formula for I- and T-section curved beams. These errors are discussed in Section 9.4. Also listed in Table 9.1 are the ratios of the maximum circumferential stress $\sigma_{\theta\theta(st)}$ given by the straight-beam flexure formula (Eq. 7.1) to the value $\sigma_{\theta\theta(\mathrm{elast})}$. The straight-beam solution is appreciably in error for small values of R/h and is in error by 7% for $R/h = 5.0$; the error is nonconservative. Generally, for curved beams with R/h greater than 5.0, the straight-beam formula may be used.

As R becomes large compared to h, the right-hand term in Eq. 9.11 reduces to $-M_x y/I_x$. The negative sign results because the sign convention for positive moments for curved beams is opposite to that for straight beams (see Eq. 7.1). To prove this reduction, note that $r = R + y$. Then the term RA_m in Eq. 9.11 may be written as

$$RA_m = \int\left(\frac{R}{R+y} + 1 - 1\right)dA = A - \int\frac{y}{R+y}\,dA \qquad \text{(a)}$$

Hence, the denominator of the right-hand term in Eq. 9.11 becomes, for $R/h \to \infty$,

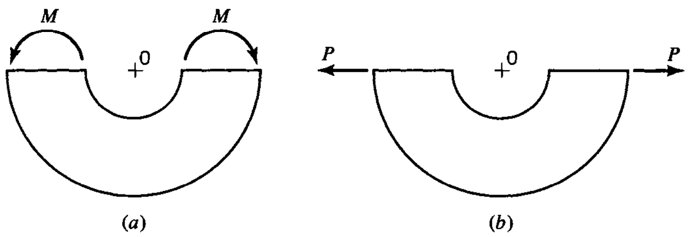


FIGURE 9.4 Types of curved beam loadings. (*a*) Pure bending. (*b*) Shear loading.

TABLE 9.1 Ratios of the Maximum Circumferential Stress in Rectangular Section Curved Beams as Computed by Elasticity Theory, the Curved Beam Formula, and the Flexure Formula

	Pure bending		Shear loading	
$\frac{R}{h}$	$\frac{\sigma_{\theta\theta(\mathrm{CB})}}{\sigma_{\theta\theta(\mathrm{elast})}}$	$\frac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(\mathrm{elast})}}$	$\frac{\sigma_{\theta\theta(\mathrm{CB})}}{\sigma_{\theta\theta(\mathrm{elast})}}$	$\frac{\sigma_{\theta\theta(st)}}{\sigma_{\theta\theta(\mathrm{elast})}}$
0.65	1.046	0.439	0.855	0.407
0.75	1.012	0.526	0.898	0.511
1.0	0.997	0.654	0.946	0.653
1.5	0.996	0.774	0.977	0.776
2.0	0.997	0.831	0.987	0.834
3.0	0.999	0.888	0.994	0.890
5.0	0.999	0.933	0.998	0.934

$$
\begin{aligned}
Ar(RA_m - A) &= -A\int\left(\frac{Ry}{R+y} + y - y\right)dA - Ay\int\frac{y}{R+y}\,dA \\
&= \frac{A}{R}\int\frac{y^2}{1+(y/R)}\,dA - A\int y\,dA - \frac{Ay}{R}\int\frac{y}{1+(y/R)}\,dA \\
&= \frac{AI_x}{R}
\end{aligned}
\tag{b}
$$

since as $R/h \to \infty$, then $y/R \to 0$, $1 + y/R \to 1$, $\int[y^2\,dA/(1 + y/R)] \to I_x$, and $\int[y\,dA/(1 + y/R)] \to \int y\,dA = 0$. The right-hand term in Eq. 9.11 then simplifies to

$$
\begin{aligned}
\frac{M_xR}{AI_x}(A - RA_m - yA_m) &= \frac{M_xR}{AI_x}\left(\int\frac{y/R}{1+(y/R)}\,dA - \frac{y}{R}\int\frac{dA}{1+(y/R)}\right) \\
&= -\frac{M_xy}{I_x}
\end{aligned}
\tag{c}
$$

The curved beam formula (Eq. 9.11) requires that A_m, defined by Eq. 9.8, be calculated for cross sections of various shapes. The number of significant digits retained in calculating A_m must be greater than that required for $\sigma_{\theta\theta}$ since RA_m approaches the value of A as R/h becomes large [see Eq. (a) above]. Explicit formulas for A, A_m, and R for several curved beam cross-sectional areas are listed in Table 9.2. Often, the cross section of a

TABLE 9.2 Expressions for A, R, and $A_m = \int \frac{dA}{r}$

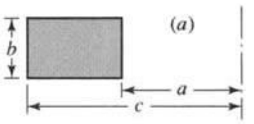
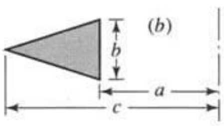
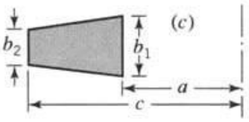
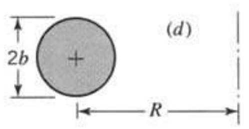
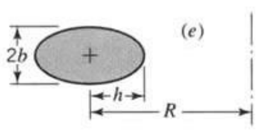
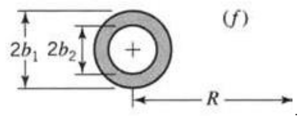
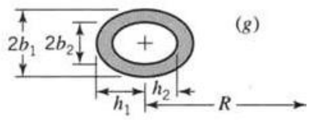
	$A = b(c-a)$; $R = \frac{a+c}{2}$ $A_m = b\ln\frac{c}{a}$
	$A = \frac{b}{2}(c-a)$; $R = \frac{2a+c}{3}$ $A_m = \frac{bc}{c-a}\ln\frac{c}{a} - b$
	$A = \frac{b_1+b_2}{2}(c-a)$; $R = \frac{a(2b_1+b_2)+c(b_1+2b_2)}{3(b_1+b_2)}$ $A_m = \frac{b_1c - b_2a}{c-a}\ln\frac{c}{a} - b_1 + b_2$
	$A = \pi b^2$ $A_m = 2\pi\left(R - \sqrt{R^2 - b^2}\right)$
	$A = \pi bh$ $A_m = \frac{2\pi b}{h}\left(R - \sqrt{R^2 - h^2}\right)$

TABLE 9.2 Expressions for *A*, *R*, and $A_m = \int \frac{dA}{r}$ *(continued)*



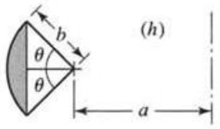
$$A = \pi\left(b_1^2 - b_2^2\right)$$

$$A_m = 2\pi\left(\sqrt{R^2 - b_2^2} - \sqrt{R^2 - b_1^2}\right)$$



$$A = \pi(b_1h_1 - b_2h_2)$$

$$A_m = 2\pi\left(\frac{b_1R}{h_1} - \frac{b_2R}{h_2} - \frac{b_1}{h_1}\sqrt{R^2 - h_1^2} + \frac{b_2}{h_2}\sqrt{R^2 - h_2^2}\right)$$



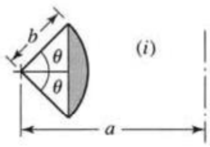
$$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; \quad R = a + \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$$

For $a > b$,

$$A_m = 2a\theta - 2b\sin\theta - \pi\sqrt{a^2 - b^2} + 2\sqrt{a^2 - b^2}\sin^{-1}\left(\frac{b + a\cos\theta}{a + b\cos\theta}\right)$$

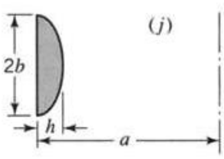
For $b > a$,

$$A_m = 2a\theta - 2b\sin\theta + 2\sqrt{b^2 - a^2}\ln\left(\frac{b + a\cos\theta + \sqrt{b^2 - a^2}\sin\theta}{a + b\cos\theta}\right)$$



$$A = b^2\theta - \frac{b^2}{2}\sin 2\theta; \quad R = a - \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$$

$$A_m = 2a\theta + 2b\sin\theta - \pi\sqrt{a^2 - b^2} - 2\sqrt{a^2 - b^2}\sin^{-1}\left(\frac{b - a\cos\theta}{a - b\cos\theta}\right)$$



$$A = \frac{\pi bh}{2}; \quad R = a - \frac{4h}{3\pi}$$

$$A_m = 2b + \frac{\pi b}{h}\left(a - \sqrt{a^2 - h^2}\right) - \frac{2b}{h}\sqrt{a^2 - h^2}\sin^{-1}\left(\frac{h}{a}\right)$$

curved beam is composed of two or more of the fundamental areas listed in Table 9.2. The values of A, A_m, and R for the composite area are given by summation. Thus, for composite cross sections,

$$A = \sum_{i=1}^{n} A_i \tag{9.12}$$

$$A_m = \sum_{i=1}^{n} A_{mi} \tag{9.13}$$

$$R = \frac{\sum_{i=1}^{n} R_i A_i}{\sum_{i=1}^{n} A_i} \tag{9.14}$$

where n is the number of fundamental areas that form the composite area.

9.2.1 Location of Neutral Axis of Cross Section

The neutral axis of bending of the cross section is defined by the condition $\sigma_{\theta\theta} = 0$. The neutral axis is located at distance R_n from the center of curvature. The distance R_n is obtained from Eq. 9.11 with the condition that $\sigma_{\theta\theta} = 0$ on the neutral surface $r = R_n$. Thus, Eq. 9.11 yields

$$R_n = \frac{AM_x}{A_m M_x + N(A - RA_m)} \tag{9.15}$$

For pure bending, $N = 0$, and then Eq. 9.15 yields

$$R_n = \frac{A}{A_m} \tag{9.16}$$

EXAMPLE 9.1
Stress in Curved Beam Portion of a Frame

The frame shown in Figure E9.1 has a 50 mm by 50 mm square cross section. The load P is located 100 mm from the center of curvature of the curved beam portion of the frame. The radius of curvature of the inner surface of the curved beam is a = 30 mm. For P = 9.50 kN, determine the values for the maximum tensile and compressive stresses in the frame.

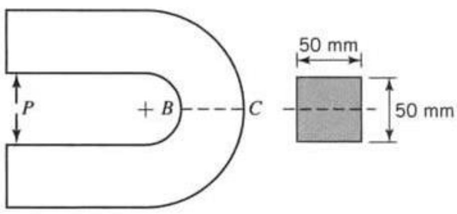


FIGURE E9.1

Solution

The circumferential stresses $\sigma_{\theta\theta}$ are calculated using Eq. 9.11. Required values for A, A_m, and R for the curved beam are calculated using the equations in row (a) of Table 9.2. For the curved beam a = 30 mm and c = 80 mm. Therefore,

$$A = b(c-a) = 50(80-30) = 2500 \text{ mm}^2$$

$$A_m = b \ln\frac{c}{a} = 50 \ln\frac{80}{30} = 49.04$$

$$R = \frac{a+c}{2} = \frac{30+80}{2} = 55 \text{ mm}$$

Hence, the maximum tensile stress is (at point B)

$$\sigma_{\theta\theta B} = \frac{P}{A} + \frac{M_x(A - rA_m)}{Ar(RA_m - A)} = \frac{9500}{2500} + \frac{155(9500)[2500 - 30(49.04)]}{2500(30)[55(49.04) - 2500]}$$

$$= 106.2 \text{ MPa}$$

The maximum compressive stress is (at point C)

$$\sigma_{\theta\theta C} = \frac{9500}{2500} + \frac{155(9500)[2500 - 80(49.04)]}{2500(80)[55(49.04) - 2500]} = -49.3 \text{ MPa}$$

EXAMPLE 9.2 ***Semicircular Aircraft Beam***

In a test of a semicircular aircraft fuselage beam, the beam is subjected to an end load $P = 300$ N that acts at the centroid of the beam cross section (Figure E9.2*a*).

(a) Using Eq. 9.11, determine the normal stress $\sigma_{\theta\theta}$ that acts on the section AB as a function of radius r and angle θ, where, by Figure E9.2*a*, $1.47 \text{ m} \le r \le 1.53$ m and $0 \le \theta \le \pi$.

(b) Determine the value of θ for which the stress $\sigma_{\theta\theta}$ is maximum.

(c) For the value of θ obtained in part (b), determine the maximum tensile and compressive stresses and their locations.

(d) Determine the maximum tensile and compressive stresses acting on the section at $\theta = \pi/2$.

(e) Compare the results obtained in parts (c) and (d) to those obtained using straight-beam theory, where $\sigma_{\theta\theta} = -My/I$.

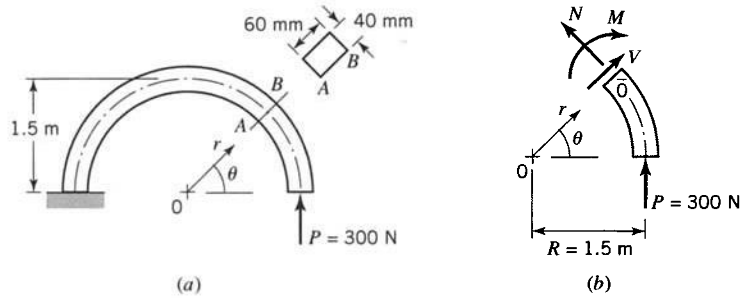


FIGURE E9.2

Solution

(a) Consider the free-body diagram of the beam segment $0 < \theta < \pi$ (Figure E9.2*b*), where N, V, and M are the normal force, the shear force, and the bending moment acting on the section at θ, respectively. By Figure E9.2*b*, we have

$$\sum F_r = V + P\sin\theta = 0$$
$$\sum F_\theta = N + P\cos\theta = 0$$
$$\circlearrowleft\!+ \quad \sum M_{\bar{0}} = PR(1 - \cos\theta) - M = 0$$

or

$$\begin{aligned} V &= -P\sin\theta = -300\sin\theta \text{ [N]} \\ N &= -P\cos\theta = -300\cos\theta \text{ [N]} \\ M &= PR(1-\cos\theta) = 450(1-\cos\theta) \text{ [N}\cdot\text{m]} \end{aligned} \tag{a}$$

For the cross section, by Figure E9.2*b* and Table 9.2,

$$\begin{aligned} A &= b(c-a) = 0.04(1.53 - 1.47) = 0.0024 \text{ m}^2 \\ A_m &= b\ln\frac{c}{a} = 0.04\ln\frac{1.53}{1.47} = 0.00160021 \text{ m} \\ R &= 1.5 \text{ m} \end{aligned} \tag{b}$$

Note that the number of digits of precision shown for A_m is required in Eq. 9.11. Now, by Eqs. (a), (b), and 9.11, we have

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)}$$

$$\sigma_{\theta\theta} = -125\cos\theta + \left(\frac{14.2857 - 9.5250r}{r}\right)(1 - \cos\theta) \times 10^5 \text{ [kPa]} \tag{c}$$

(b) For maximum (or minimum) $\sigma_{\theta\theta}$,

$$\frac{d\sigma_{\theta\theta}}{d\theta} = \left[125 + \left(\frac{14.2857 - 9.5250r}{r}\right) \times 10^5\right]\sin\theta = 0$$

Hence, $\sigma_{\theta\theta}$ is minimum at $\theta = 0$ and it is maximum at $\theta = \pi$, with values given by Eq. (c).

(c) From Eq. (c), the maximum tensile and compressive stresses at $\theta = \pi$ are as follows:

For $r = 1.47$ m, the tensile stress at A is

$$\sigma_{\theta\theta} = 125 + 38{,}633 = 38{,}758 \text{ kPa} = 38.76 \text{ MPa} \tag{d}$$

For $r = 1.53$ m, the compressive stress at B is

$$\sigma_{\theta\theta} = 125 - 37{,}588 = -37{,}463 \text{ kPa} = -37.46 \text{ MPa} \tag{e}$$

(d) By Eq. (c), with $\theta = \pi/2$,

For $r = 1.47$ m, the tensile stress at A is

$$\sigma_{\theta\theta} = 0 + 19{,}316 \text{ kPa} = 19.32 \text{ MPa} \tag{f}$$

For $r = 1.53$ m, the compressive stress at B is

$$\sigma_{\theta\theta} = 0 - 18{,}794 \text{ kPa} = -18.79 \text{ MPa} \tag{g}$$

(e) Using straight-beam theory, we have

$$\sigma_{\theta\theta} = -\frac{My}{I} \tag{h}$$

where, by Figure E9.2*a*,

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(0.04)(0.06)^3 = 7.2 \times 10^{-7} \text{ m}^4$$

and for $\theta = \pi$,

$$M = 2PR = 2(300)(1.5) = 900 \text{ N} \bullet \text{m}$$

Hence, by Eq. (h),

$$\sigma_{\theta\theta} = -(1.25 \times 10^9)y \tag{i}$$

For $y = -0.03$ m (point A in Figure E9.2*a*), Eq. (i) yields $\sigma_{\theta\theta} = 37.50$ MPa, compared to $\sigma_{\theta\theta} = 38.76$ MPa in part (c). For $y = +0.03$ m (point B in Figure E9.2*a*), Eq. (i) yields $\sigma_{\theta\theta} = -37.50$ MPa, compared to $\sigma_{\theta\theta} = 37.46$ MPa in part (c).

For $\theta = \pi/2$,

$$M = PR = (300)(1.5) = 450 \text{ N} \bullet \text{m}$$

and then by Eq. (h),

$$\sigma_{\theta\theta} = -(6.25 \times 10^8)y \tag{j}$$

For $y = -0.03$ m (point A), Eq. (j) yields $\sigma_{\theta\theta} = 18.75$ MPa, compared to $\sigma_{\theta\theta} = 19.32$ MPa in part (d). For $y = +0.03$ m (point B), Eq. (j) yields $\sigma_{\theta\theta} = -18.75$ MPa, compared to $\sigma_{\theta\theta} = 18.79$ MPa in part (d).

EXAMPLE 9.3
Stresses in a Crane Hook

Section BC is the critically stressed section of a crane hook (Figure E9.3*a*). For a large number of manufactured crane hooks, the critical section BC can be closely approximated by a trapezoidal area with half of an ellipse at the inner radius and an arc of a circle at the outer radius. Such a section is shown in Figure E9.3*b*, which includes dimensions for the critical cross section. The crane hook is made of a ductile steel that has a yield stress of Y = 500 MPa. Assuming that the crane hook is designed with a factor of safety of SF = 2.00 against initiation of yielding, determine the maximum load P that can be carried by the crane hook.

Note: An efficient algorithm to analyze crane hooks has been developed by Wang (1985).

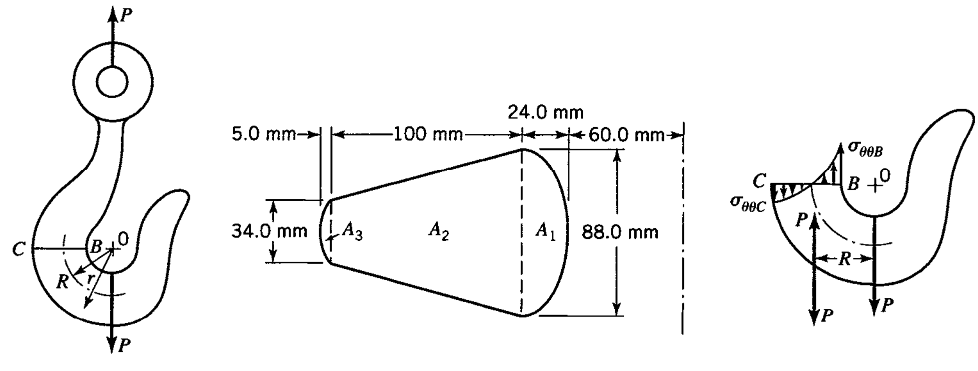


FIGURE E9.3 Crane hook.

Solution

The circumferential stresses $\sigma_{\theta\theta}$ are calculated using Eq. 9.11. To calculate values of A, R, and A_m for the curved beam cross section, we divide the cross section into basic areas A_1, A_2, and A_3 (Figure E9.3*b*).

For area A_1, a = 84 mm. Substituting this dimension along with other given dimensions into Table 9.2, row (j), we find

$$A_1 = 1658.76 \text{ mm}^2, \quad R_1 = 73.81 \text{ mm}, \quad A_{m1} = 22.64 \text{ mm} \qquad \text{(a)}$$

For the trapezoidal area A_2, a = 60 + 24 = 84 mm and $c = a$ + 100 = 184 mm. Substituting these dimensions along with other given dimensions into Table 9.2, row (c), we find

$$A_2 = 6100.00 \text{ mm}^2, \quad R_2 = 126.62 \text{ mm}, \quad A_{m2} = 50.57 \text{ mm} \qquad \text{(b)}$$

For area A_3, θ = 0.5721 rad, b = 31.40 mm, and a = 157.60 mm. When these values are substituted into Table 9.2, row (h), we obtain

$$A_3 = 115.27 \text{ mm}^2, \quad R_3 = 186.01 \text{ mm}, \quad A_{m3} = 0.62 \text{ mm} \qquad \text{(c)}$$

Substituting values of A_i, R_i, and A_{mi} from Eqs. (a)–(c) into Eqs. 9.12–9.14, we calculate

$$\begin{aligned} A &= 6100.00 + 115.27 + 1658.76 = 7874.03 \text{ mm}^2 \\ A_m &= 50.57 + 0.62 + 22.64 = 73.83 \text{ mm} \\ R &= \frac{6100.00(126.62) + 115.27(186.01) + 1658.76(73.81)}{7874.03} \\ &= 116.37 \text{ mm} \end{aligned}$$

As indicated in Figure E9.3*c*, the circumferential stress distribution $\sigma_{\theta\theta}$ is due to the normal load $N = P$ and moment $M_x = PR$. The maximum tension and compression values of $\sigma_{\theta\theta}$ occur at points B and C, respectively. For points B and C, by Figure E9.3*b*, we find

$$r_B = 60 \text{ mm}$$

$$r_C = 60 + 24 + 100 + 5 = 189 \text{ mm}$$

Substituting the required values into Eq. 9.11, we find

$$\sigma_{\theta\theta B} = \frac{P}{7874.03} + \frac{116.37P[7874.03 - 60(73.83)]}{7874.03(60)[116.37(73.83) - 7874.03]}$$

$$= 0.000127P + 0.001182P$$

$$= 0.001309P \quad \text{(tension)}$$

$$\sigma_{\theta\theta C} = \frac{P}{7874.03} + \frac{116.37P[7874.03 - 189(73.83)]}{7874.03(189)[116.37(73.83) - 7874.03]}$$

$$= 0.000127P - 0.000662P$$

$$= -0.000535P \quad \text{(compression)}$$

Since the absolute magnitude of $\sigma_{\theta\theta B}$ is greater than $\sigma_{\theta\theta C}$, initiation of yield occurs when $\sigma_{\theta\theta B}$ equals the yield stress Y. The corresponding value of the failure load (P_f) is the load at which yield occurs. Dividing the failure load $P_f = Y/(0.001309)$ by the factor of safety $SF = 2.00$, we obtain the design load P; namely,

$$P = \frac{500}{2.00(0.001309)} = 190{,}900 \text{ N}$$

EXAMPLE 9.4
Proof Test of a Crane Hook

To proof test a crane hook an engineer applies a load P to the hook through a pin (Figure E9.4*a*). Assume that the pin exerts a pressure $p \sin\theta$ [N/mm²] at radius r_i for $0 \le \theta \le \pi$, where p is a constant. The hook has a uniform rectangular cross section of thickness t.

(a) Determine the circumferential stress $\sigma_{\theta\theta}$ as a function of P, r_o, r_i, r, and θ.

(b) For $r_i = 60$ mm, $r_o = 180$ mm, and $t = 50$ mm, determine the maximum tensile and compressive stresses on the cross section at $\theta = \pi/2$ and $\theta = \pi$ in terms of P.

(c) If the maximum allowable tensile stress is $\sigma_{\theta\theta} = 340$ MPa, what is the allowable load P for a safety factor of 2.2?

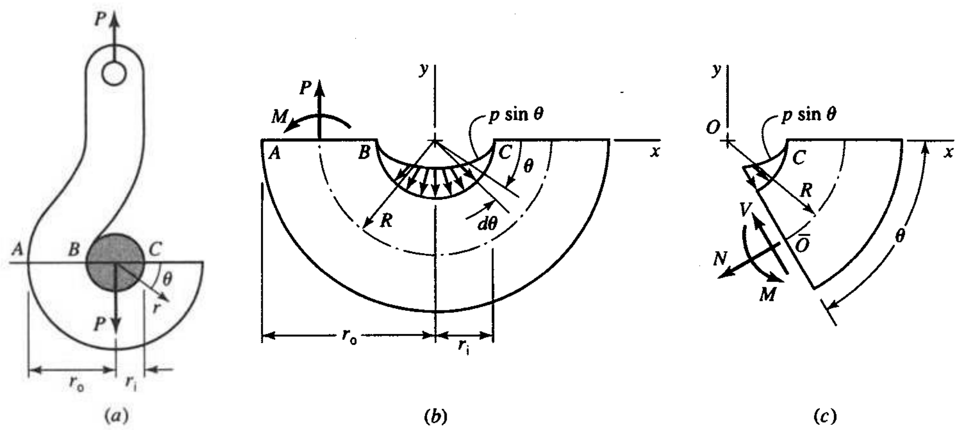


FIGURE E9.4

Solution

(a) Consider the free-body diagram of the hook segment *ABC* (Figure E9.4*b*). Summing forces in the *y* direction, we have

$$\sum F_y = P - \int_0^{\pi} [(p\sin\phi)(r_i\, d\phi)t]\sin\phi = 0$$

or

$$p = \frac{2P}{\pi r_i t} \qquad \text{(a)}$$

Next consider the free-body diagram of an element of the hook. By Figure E9.4*c* we have for equilibrium in the *x* direction

$$\sum F_x = -N\sin\theta - V\cos\theta + pr_i t\int_0^{\theta} \sin\phi\ \cos\phi\ d\phi = 0$$

or

$$N\sin\theta + V\cos\theta = \frac{1}{4}pr_i t(1-\cos 2\theta) \qquad \text{(b)}$$

For equilibrium in the *y* direction, we have

$$\sum F_y = -N\cos\theta + V\sin\theta - pr_i t\int_0^{\theta} \sin\phi\ \sin\phi\ d\phi = 0$$

or

$$N\cos\theta - V\sin\theta = -\frac{1}{4}pr_i t(2\theta - \sin 2\theta) \qquad \text{(c)}$$

For equilibrium of moments

$$\circlearrowleft\!+\ \sum M_0 = M - NR = 0$$

or

$$M = NR \qquad \text{(d)}$$

The solution of Eqs. (b), (c), and (d) is

$$N = \frac{1}{2}pr_i t(\sin\theta - \theta\cos\theta) \qquad \text{(e)}$$

$$V = \frac{1}{2}pr_i t(\theta\sin\theta) \qquad \text{(f)}$$

$$M = \frac{1}{2}pr_i Rt(\sin\theta - \theta\cos\theta) \qquad \text{(g)}$$

By Eqs. (e), (g), and 9.11,

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M(A - rA_m)}{Ar(RA_m - A)} \qquad \text{(h)}$$

where

$$A = (r_o - r_i)t$$
$$A_m = t \ln\frac{r_o}{r_i} \qquad \text{(i)}$$
$$R = \frac{1}{2}(r_o + r_i)$$

and by Eqs. (a), (e), and (g),

$$N = \frac{P}{\pi}(\sin\theta - \theta\cos\theta)$$
$$M = \frac{PR}{\pi}(\sin\theta - \theta\cos\theta) \qquad \text{(j)}$$

Hence, by Eqs. (h), (i), and (j),

$$\sigma_{\theta\theta} = \frac{P(\sin\theta - \theta\cos\theta)}{\pi(r_o - r_i)t}\left\{1 + \frac{(r_o + r_i)\left(r_o - r_i - r\ln\dfrac{r_o}{r_i}\right)}{r\left[(r_o + r_i)\ln\dfrac{r_o}{r_i} - 2(r_o - r_i)\right]}\right\} \qquad \text{(k)}$$

(b) For $r_i = 60$ mm, $r_o = 180$ mm, and $t = 50$ mm, Eq. (k) yields

$$\sigma_{\theta\theta} = P(\sin\theta - \theta\cos\theta)\left(\frac{0.06456}{r} - 0.0005372\right) \qquad \text{(l)}$$

For $\theta = \pi/2$, Eq. (l) yields

$$\sigma_{\theta\theta} = P\left(\frac{0.06456}{r} - 0.0005372\right)$$

For maximum tensile stress, $r = r_i = 60$ mm, at which

$$(\sigma_{\theta\theta})_{max} = 0.000539P \qquad \text{(m)}$$

For maximum compressive stress, $r = r_o = 180$ mm, at which

$$(\sigma_{\theta\theta})_{max} = -0.000179P \qquad \text{(n)}$$

For $\theta = \pi$, Eq. (l) yields

$$\sigma_{\theta\theta} = P\left(\frac{0.2028}{r} - 0.001690\right)$$

For maximum tensile stress, $r = r_i = 60$ mm, at which

$$(\sigma_{\theta\theta})_{max} = 0.00169P \qquad \text{(o)}$$

For maximum compressive stress, $r = r_o = 180$ mm, at which

$$(\sigma_{\theta\theta})_{max} = -0.000563P \qquad \text{(p)}$$

(c) For a maximum allowable tensile stress of 340 MPa and a safety factor of 2.2, Eq. (o) yields

$$\frac{340}{2.2} = 0.00169P$$

or the maximum allowable load is $P = 91{,}447$ N = 91.45 kN.

RADIAL STRESSES IN CURVED BEAMS

The curved beam formula for circumferential stress $\sigma_{\theta\theta}$ (Eq. 9.11) is based on the assumption that the effect of radial stress is small. This assumption is accurate for curved beams with circular, rectangular, or trapezoidal cross sections, that is, cross sections that do not possess thin webs. However, in curved beams with cross sections in the form of an H, T, or I, the webs may be so thin that the maximum radial stress in the web may exceed the maximum circumferential stress. Also, although the radial stress is usually small, it may be significant relative to radial strength, for example, when anisotropic materials, such as wood, are formed into curved beams. The beam should be designed to take such conditions into account.

To illustrate these remarks, we consider the tensile radial stress, resulting from a positive moment, that occurs in a curved beam at radius r from the center of curvature 0 of the beam (Figure 9.5*a*). Consider equilibrium of the element $BDGF$ of the beam, shown enlarged in the free-body diagram in Figure 9.5*c*. The faces BD and GF, which subtend the infinitesimal angle $d\theta$, have the area A' shown shaded in Figure 9.5*b*. The distribution of $\sigma_{\theta\theta}$ on each of these areas produces a resultant circumferential force T (Figure 9.5*c*) given by the expression

$$T = \int_a^r \sigma_{\theta\theta}\, dA \tag{9.17}$$

The components of the circumferential forces along line $0L$ are balanced by the radial stress σ_{rr} acting on the area $tr\, d\theta$, where t is the thickness of the cross section at the distance r from the center of curvature 0 (Figure 9.5*b*). Thus for equilibrium in the radial direction along $0L$,

$$\sum F_r = 0 = \sigma_{rr} tr\, d\theta - 2T \sin(d\theta/2) = (\sigma_{rr} tr - T)\, d\theta$$

since for infinitesimal angle $d\theta/2$, $\sin(d\theta/2) = d\theta/2$. Therefore, the tensile stress resulting from the positive moment is

$$\sigma_{rr} = \frac{T}{tr} \tag{9.18}$$

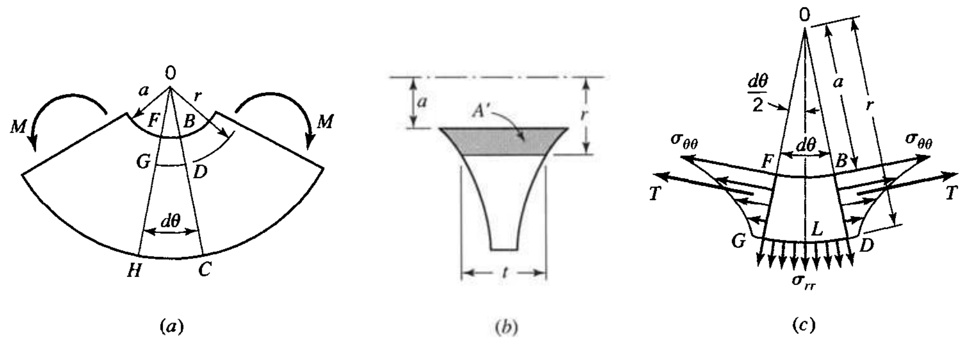


FIGURE 9.5 Radial stress in a curved beam. (*a*) Side view. (*b*) Cross-sectional shape. (*c*) Element *BDGF*.

The force T is obtained by substitution of Eq. 9.11 into Eq. 9.17. Thus,

$$T = \frac{N}{A}\int_a^r dA + \frac{M_x}{RA_m - A}\int_a^r \frac{dA}{r} - \frac{M_x A_m}{A(RA_m - A)}\int_a^r dA$$
$$T = \frac{A'}{A}N + \frac{AA'_m - A'A_m}{A(RA_m - A)}M_x \qquad (9.19)$$

where

$$A'_m = \int_a^r \frac{dA}{r} \quad \text{and} \quad A' = \int_a^r dA \qquad (9.20)$$

Substitution of Eq. 9.19 into Eq. 9.18 yields the relation for the radial stress. For rectangular cross section curved beams subjected to shear loading (Figure 9.4*b*), a comparison of the resulting approximate solution with the elasticity solution indicates that the approximate solution is conservative. Furthermore, for such beams it remains conservative to within 6% for values of $R/h > 1.0$ even if the term involving N in Eq. 9.19 is discarded. Consequently, if we retain only the moment term in Eq. 9.19, the expression for the radial stress may be approximated by the formula

$$\sigma_{rr} = \frac{AA'_m - A'A_m}{trA(RA_m - A)}M_x \qquad (9.21)$$

to within 6% of the elasticity solution for rectangular cross section curved beams subjected to shear loading (Figure 9.4*b*).

9.3.1 Curved Beams Made from Anisotropic Materials

Typically, the radial stresses developed in curved beams of *stocky* (rectangular, circular, etc.) cross sections are small enough that they can be neglected in analysis and design. However, some anisotropic materials may have low strength in the radial direction. Such materials include fiber-reinforced composites (fiberglass) and wood. For these materials, the relatively small radial stress developed in a curved beam may control the design of the beam owing to the corresponding relatively low strength of the material in the radial direction. Hence, it may be important to properly account for radial stresses in curved beams of certain materials.

EXAMPLE 9.5
Radial Stress in a T-Section

The curved beam in Figure E9.5 is subjected to a load $P = 120$ kN. The dimensions of section BC are also shown. Determine the circumferential stress at B and radial stress at the junction of the flange and web at section BC.

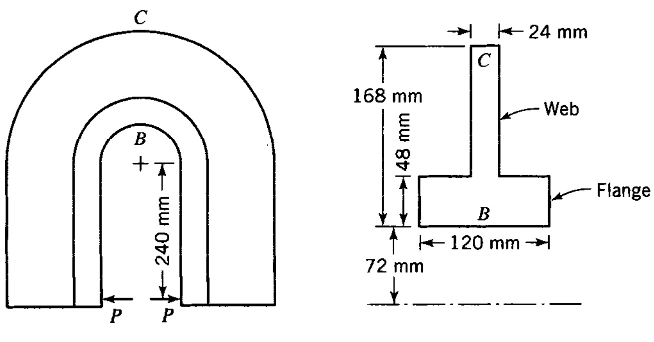


FIGURE E9.5

Solution

The magnitudes of A, A_m, and R are given by Eqs. 9.12, 9.13, and 9.14, respectively. They are

$$A = 48(120) + 120(24) = 8640 \text{ mm}^2$$

$$R = \frac{48(120)(96) + 120(24)(180)}{8640} = 124.0 \text{ mm}$$

$$A_m = 120 \ln\frac{120}{72} + 24 \ln\frac{240}{120} = 77.93 \text{ mm}$$

The circumferential stress is given by Eq. 9.11. It is

$$\sigma_{\theta\theta B} = \frac{120{,}000}{8640} + \frac{364.0(120{,}000)[8640 - 72(77.93)]}{8640(72)[124.0(77.93) - 8640]}$$

$$= 13.9 + 207.8 = 221.7 \text{ MPa}$$

The radial stress at the junction of the flange and web is given by Eq. 9.21, with $r = 120$ mm and $t = 24$ mm. Magnitudes of A' and A'_m are

$$A' = 48(120) = 5760 \text{ mm}^2$$

$$A'_m = 120 \ln\frac{120}{72} = 61.30 \text{ mm}$$

Substitution of these values into Eq. 9.21, which neglects the effect of N, gives

$$\sigma_{rr} = \frac{364.0(120{,}000)[8640(61.30) - 5760(77.93)]}{24(120)(8640)[124.0(77.93) - 8640]} = 138.5 \text{ MPa}$$

Hence, the magnitude of this radial stress is appreciably less than the maximum circumferential stress ($|\sigma_{\theta\theta B}| > |\sigma_{\theta\theta C}|$) and may not be of concern for the design engineer.

EXAMPLE 9.6 *Radial Stress in an I-Section*

The curved section of the frame of a press is subjected to a positive moment $M_0 = 96$ kN • m and a shear load $P = 120$ kN (Figure E9.6*a*). The dimensions of section BC are shown in Figure E9.6*b*. Determine the circumferential stress $\sigma_{\theta\theta}$ at point B and the radial stress σ_{rr} at points B' and C' of section BC. Include the effects of traction N.

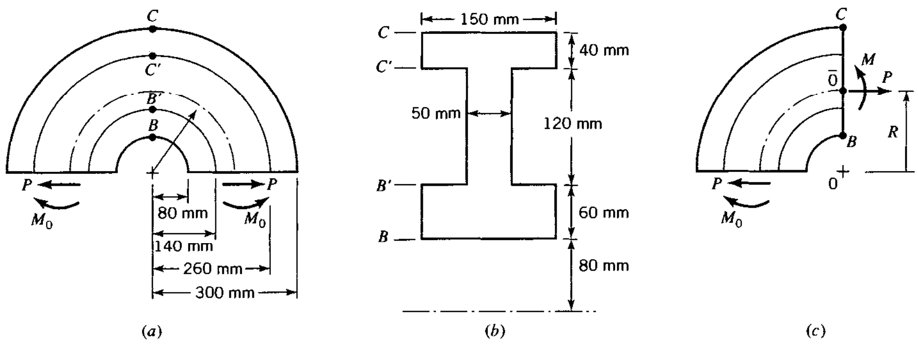


FIGURE E9.6

Solution

The magnitudes of A, A_m, and R are given by Eqs. 9.12, 9.13, and 9.14. They are

$$A = 150(60) + 50(120) + 150(40) = 21{,}000 \text{ mm}^2$$

$$A_m = 150 \ln\frac{140}{80} + 50 \ln\frac{260}{140} + 150 \ln\frac{300}{260} = 136.360 \text{ mm} \quad \text{(a)}$$

$$R = \frac{150(60)110 + 50(120)200 + 150(40)280}{21{,}000} = 184.286 \text{ mm}$$

By Figure E9.6*c*,

$$\circlearrowleft\!+ \quad \sum M_{\bar{0}} = M - M_0 - PR = 0$$

or

$$M = 96{,}000{,}000 + 120{,}000(184.286) = 118.1 \times 10^6 \text{ N} \bullet \text{mm}$$

Then, by Eq. 9.11 with $r = 80$ mm, the circumferential stress at b is

$$\sigma_{\theta\theta} = \frac{120{,}000}{21{,}000} + \frac{118{,}100{,}000\,[21{,}000 - 80(136.360)]}{21{,}000(80)[184.286(136.360) - 21{,}000]}$$
$$= 5.71 + 171.80 = 177.51 \text{ MPa}$$

To find the radial stress σ_{rr} at the junction of the flange and web (point B'), we require the geometric terms A' and A'_m. By Eq. 9.20,

$$A' = 150(60) = 9000 \text{ mm}^2$$
$$A'_m = \int_{80}^{140} 150\frac{dr}{r} = 150 \ln\frac{140}{80} = 83.94 \text{ mm} \qquad \text{(b)}$$

With the values in Eq. (b), $r = 140$ mm, and $t = 50$ mm, Eqs. 9.18 and 9.19 yield

$$\sigma_{rr} = \frac{A'}{A}\frac{N}{tr} + \frac{AA'_m - A'A_m}{trA(RA_m - A)}M$$
$$= \frac{9000}{21{,}000}\frac{120{,}000}{50(140)} + \frac{[21{,}000(83.94) - 9000(136.360)]}{50(140)(21{,}000)[184.286(136.360) - 21{,}000]}(118.1 \times 10^6)$$
$$= 7.347 + 104.189 = 111.54 \text{ MPa}$$

Here we see that the effect of N represents $(7.347/111.54) \times 100\% = 6.59\%$ of the total σ_{rr} at B'.

Similarly, for the radial stress at point C', where $r = 260$ mm and $t = 50$ mm, the geometric terms A' and A'_m are

$$A' = 150(60) + 50(120) = 15{,}000 \text{ mm}^2$$
$$A'_m = \int_{80}^{140} 150\frac{dr}{r} + \int_{140}^{260} 50\frac{dr}{r} = 114.89 \text{ mm} \qquad \text{(c)}$$

Then, by Eqs. 9.18, 9.19, and (c), we have

$$\sigma_{rr} = \frac{15{,}000}{21{,}000}\frac{120{,}000}{50(260)} + \frac{[21{,}000(114.89) - 15{,}000(136.360)]}{50(260)(21{,}000)[184.286(136.360) - 21{,}000]}(118.1 \times 10^6)$$
$$= 6.59 + 38.48 = 45.07 \text{ MPa}$$

At C', the effect of N represents 14.6% of the total radial stress. In either case (point B' or C'), σ_{rr} is considerably less than $\sigma_{\theta\theta} = 177.54$ MPa.

EXAMPLE 9.7
Radial Stress in Glulam Beam

A glued laminated timber (glulam) beam is used in a roof system. The beam has a simple span of 15 m and the middle half of the beam is curved with a mean radius of 10 m. The beam depth and width are both constant: $d = 0.800$ m and $b = 0.130$ m. Dead load is 2400 N/m and snow load is 4800 N/m. The geometry of the beam and assumed loading are shown in Figure E9.7.

(a) Determine the maximum circumferential and radial stresses in the beam.

(b) Compare the maximum circumferential stress to that obtained from the straight-beam flexure formula.

(c) Compare the maximum circumferential and radial stresses to the allowable stress limits for Douglas fir: $\sigma_{\theta\theta(\text{allow})} = 15.8$ MPa, $\sigma_{rr(\text{allow})} = 0.119$ MPa (AITC, 1994).

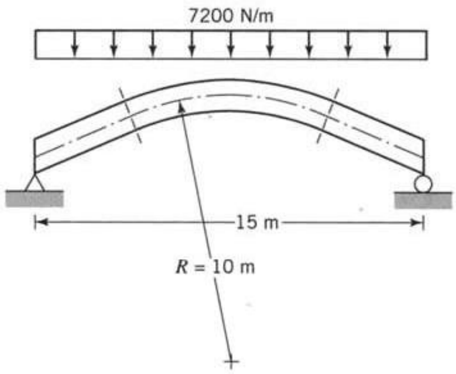


FIGURE E9.7

Solution

(a) The maximum bending moment occurs at midspan and has magnitude $M_x = wl^2/8 = 202{,}500$ N • m. Circumferential stress $\sigma_{\theta\theta}$ is calculated using Eq. 9.11. For the curved beam described,

$$a = R - \frac{d}{2} = 9.6 \text{ m}$$

$$c = R + \frac{d}{2} = 10.4 \text{ m}$$

$$A = 0.13 \times 0.80 = 0.104 \text{ m}^2$$

$$A_m = 0.13 \ln\frac{10.4}{9.6} = 0.0104056$$

The maximum circumferential stress occurs at the inner edge of the beam $r = a$. It is

$$\sigma_{\theta\theta(\max)} = \frac{M_x(A - aA_m)}{Aa(RA_m - A)} = \frac{202{,}500[0.104 - 9.6(0.0104056)]}{0.104(9.6)[10.0(0.0104056) - 0.104]} = 15.0 \text{ MPa}$$

The maximum radial stress $\sigma_{rr(\max)}$ is calculated using Eq. 9.21. However, the location at which $\sigma_{rr(\max)}$ occurs is unknown. Thus, we must maximize σ_{rr} with respect to r. For a rectangular cross section, the quantities in Eq. 9.21 are

$$t = b = \text{width of cross section}$$

$$d = c - a = \text{depth of cross section}$$

$$A = bd$$

$$A' = b(r - a)$$

$$A_m = b \ln\frac{c}{a}$$

$$A'_m = b \ln\frac{r}{a}$$

Substitution of these expressions into Eq. 9.21 gives

$$\sigma_{rr} = \frac{M_x}{b}\left[\frac{d\ln\left(\frac{r}{a}\right) - (r-a)\ln\left(\frac{c}{a}\right)}{rd\left[R\ln\left(\frac{c}{a}\right) - d\right]}\right] \tag{a}$$

Maximizing σ_{rr} with respect to r, we find that $\sigma_{rr(\max)}$ occurs at

$$r = ae^{\left(1 - \frac{a}{d}\ln\frac{c}{a}\right)} \tag{b}$$

We evaluate Eq. (b) for the particular cross section of this example to obtain r = 9.987 m. At that location, the radial stress is, by Eq. (a),

$$\sigma_{rr(\max)} = \frac{202{,}500}{0.13}\left[\frac{0.80\ln\left(\frac{9.987}{9.6}\right) - \left[(9.987-9.6)\ln\left(\frac{10.4}{9.6}\right)\right]}{9.987(0.80)\left[10.0\ln\left(\frac{10.4}{9.6}\right) - 0.80\right]}\right] \tag{c}$$

$$= 0.292 \text{ MPa}$$

An approximate formula for computing radial stress in curved beams of rectangular cross section is (AITC, 1994, p. 227)

$$\sigma_{rr} = \frac{3M}{2Rbd} \tag{d}$$

Using this expression, we determine the radial stress to be σ_{rr} = 0.292 MPa. The approximation of Eq. (d) is quite accurate in this case! In fact, for rectangular curved beams with $R/d > 3$, the error in Eq. (d) is less than 3%. However, as R/d becomes small, the error grows substantially and Eq. (d) is nonconservative.

(b) Using the curved beam formula, Eq. 9.11, we obtain the maximum circumferential stress as $\sigma_{\theta\theta(\max)}$ = 15.0 MPa. Using the straight-beam flexure formula, Eq. 7.1, with $I_x = bd^3/12$ = 0.005547 m^4, we obtain $\sigma_{\theta\theta}$ = 202,500(0.40)/0.005547 = 14.6 MPa. Thus, the straight-beam flexure formula is within 3% of the curved beam formula. One would generally consider the flexure formula adequate for this case, in which R/d = 12.5.

(c) The maximum circumferential stress is just within its limiting value; the beam is understressed just 5%. However, the maximum radial stress is 245% over its limit. It would be necessary to modify beam geometry or add mechanical reinforcement to make this design acceptable.

CORRECTION OF CIRCUMFERENTIAL STRESSES IN CURVED BEAMS HAVING I, T, OR SIMILAR CROSS SECTIONS

If the curved beam formula is used to calculate circumferential stresses in curved beams having thin flanges, the computed stresses are considerably in error and the error is nonconservative. The error arises because the radial forces developed in the curved beam causes the tips of the flanges to deflect radially, thereby distorting the cross section of the curved beam. The resulting effect is to decrease the stiffness of the curved beam, to decrease the circumferential stresses in the tips of the flanges, and to increase the circumferential stresses in the flanges near the web.

Consider a short length of a thin-flanged I-section curved beam included between faces BC and FH that form an infinitesimal angle $d\theta$ as indicated in Figure 9.6*a*. If the curved beam is subjected to a positive moment M_x, the circumferential stress distribution results in a tensile force T acting on the inner flange and a compressive force C acting on the outer flange, as shown. The components of these forces in the radial direction are $T\,d\theta$ and $C\,d\theta$. If the cross section of the curved beam did not distort, these forces would be uniformly distributed along each flange, as indicated in Figure 9.6*b*. However, the two portions of the tension and compression flanges act as cantilever beams fixed at the web. The resulting bending because of cantilever beam action causes the flanges to distort, as indicated in Figure 9.6*c*.

The effect of the distortion of the cross section on the circumferential stresses in the curved beam can be determined by examining the portion of the curved beam $ABCD$ in Figure 9.6*d*. Sections AC and BD are separated by angle θ in the unloaded beam. When the curved beam is subjected to a positive moment, the center of curvature moves from 0 to 0*, section AC moves to A^*C^*, section BD moves to B^*D^*, and the included angle becomes θ^*. If the cross section does not distort, the inner tension flange AB elongates to length A^*B^*. Since the tips of the inner flange move radially inward relative to the undistorted position (Figure 9.6*c*), the circumferential elongation of the tips of the inner flange is less than that indicated in Figure 9.6*d*. *Therefore,* $\sigma_{\theta\theta}$ *in the tips of the inner flange is less than that calculated using the curved beam formula.* To satisfy equilibrium, it is necessary that $\sigma_{\theta\theta}$ for the portion of the flange near the web be greater than that calculated using the curved beam formula. Now consider the outer compression flange. As indicated in Figure 9.6*d*, the outer flange shortens from CD to C^*D^* if the cross section does not distort. Because of the distortion (Figure 9.6*c*), the tips of the compressive flange move radially outward, requiring less compressive contraction. *Therefore, the magnitude of* $\sigma_{\theta\theta}$

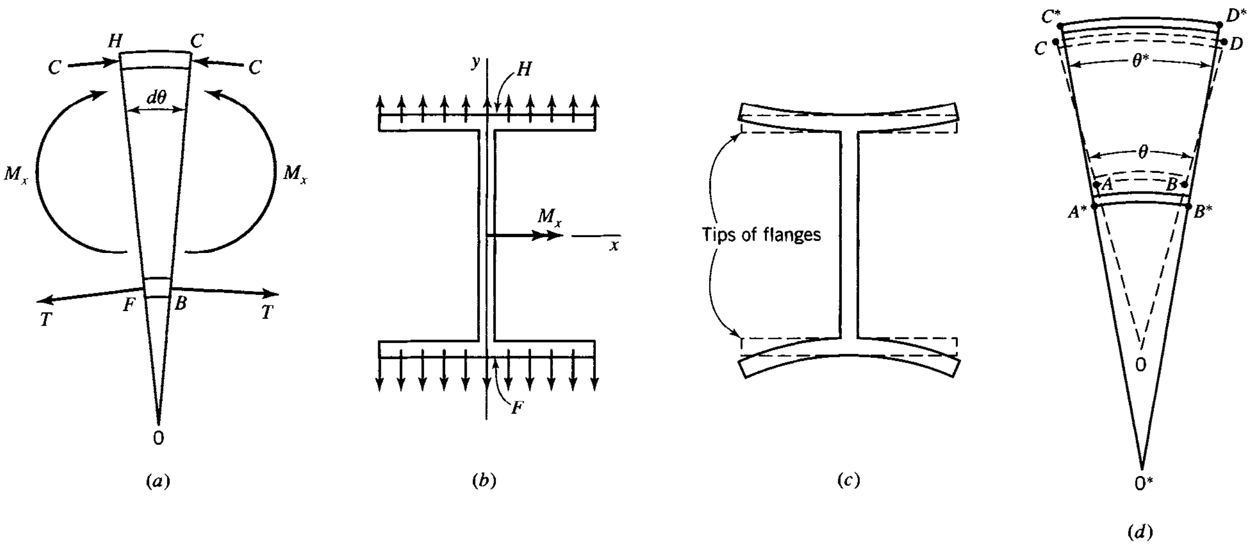


FIGURE 9.6 Distortion of cross section of an I-section curved beam.

in the tips of the compression outer flange is less than that calculated by the curved beam formula, and the magnitude of $\sigma_{\theta\theta}$ in the portion of the compression flange near the web is larger than that calculated by the curved beam formula.

The resulting circumferential stress distribution is indicated in Figure 9.7. Since in developing the curved beam formula we assume that the circumferential stress is independent of x (Figure 9.2), corrections are required if the formula is to be used in the design of curved beams having I, T, and similar cross sections. There are two approaches that can be employed in the design of these curved beams. One approach is to prevent the radial distortion of the cross section by welding radial stiffeners to the curved beams. If distortion of the cross section is prevented, the use of the curved beam formula is appropriate. A second approach, suggested by H. Bleich (1933), is discussed next.

9.4.1 Bleich's Correction Factors

Bleich reasoned that the actual maximum circumferential stresses in the tension and compression flanges for the I-section curved beam (Figure 9.8*a*) can be calculated by the curved beam formula applied to an I-section curved beam with *reduced flange widths,* as indicated in Figure 9.8*b*. By Bleich's method, if the same bending moment is applied to the two cross sections in Figure 9.8, the computed maximum circumferential tension and

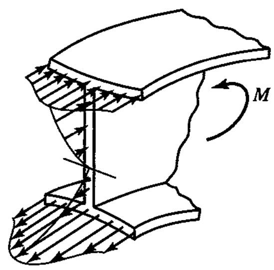


FIGURE 9.7 Stresses in I-section of curved beam.

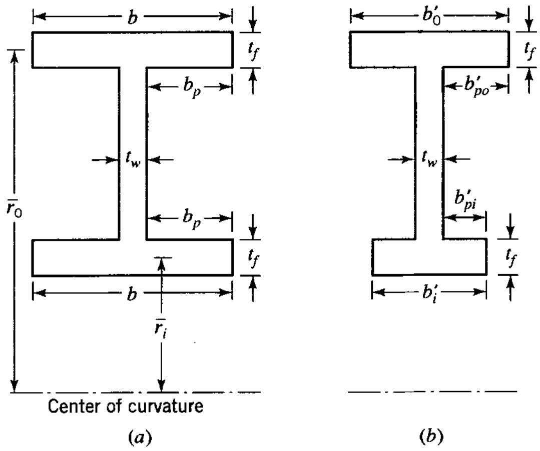


FIGURE 9.8 (*a*) Actual and (*b*) modified I-section for a curved beam.

compression stresses for the cross section shown in Figure 9.8*b*, with no distortion, are equal to the actual maximum circumferential tension and compression stresses for the cross section in Figure 9.8*a*, with distortion.

The approximate solution proposed by Bleich gives the results presented in tabular form in Table 9.3. To use the table, the ratio $b_p^2/\bar{r}t_f$ must be calculated, where

$$b_p = \text{projecting width of flange (see Figure 9.8}a\text{)}$$
$$\bar{r} = \text{radius of curvature to the center of flange}$$
$$t_f = \text{thickness of flange}$$

The reduced width b'_p of the projecting part of each flange (Figure 9.8*b*) is given by the relation

$$b'_p = \alpha b_p \tag{9.22}$$

where α is obtained from Table 9.3 for the computed value of the ratio $b_p^2/\bar{r}t_f$. The reduced width of each flange (Figure 9.8*b*) is given by

$$b' = 2b'_p + t_w \tag{9.23}$$

where t_w is the thickness of the web. When the curved beam formula (Eq. 9.11) is applied to an undistorted cross section corrected by Eq. 9.23, it predicts the maxmum circumferential stress in the actual (distorted) cross section. This maximum stress occurs at the center of the inner flange. It should be noted that the state of stress at this point in the curved beam is not uniaxial. Because of the bending of the flanges (Figure 9.6*c*), a transverse component of stress σ_{xx} (Figure 9.2) is developed; the sign of σ_{xx} is opposite to that of $\sigma_{\theta\theta(\max)}$. Bleich obtained an approximate solution for σ_{xx} for the inner flange. It is given by the relation

$$\sigma_{xx} = -\beta\bar{\sigma}_{\theta\theta} \tag{9.24}$$

where β is obtained from Table 9.3 for the computed value of the ratio $b_p^2/\bar{r}t_f$, and where $\bar{\sigma}_{\theta\theta}$ is the magnitude of the circumferential stress at midthickness of the inner flange; the value of $\bar{\sigma}_{\theta\theta}$ is calculated based on the corrected cross section.

Although Bleich's analysis was developed for curved beams with relatively thin flanges, the results agree closely with a similar solution obtained by C. G. Anderson (1950) for I-beams and box beams, in which the analysis was not restricted to thin-flanged sections. Similar analyses of tubular curved beams with circular and rectangular cross

TABLE 9.3 Table for Calculating the Effective Width and Lateral Bending Stress of Curved I- or T-Beams

$b_p^2/\bar{r}t_f$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α	0.977	0.950	0.917	0.878	0.838	0.800	0.762	0.726	0.693
β	0.580	0.836	1.056	1.238	1.382	1.495	1.577	1.636	1.677
$b_p^2/\bar{r}t_f$	1.1	1.2	1.3	1.4	1.5	2.0	3.0	4.0	5.0
α	0.663	0.636	0.611	0.589	0.569	0.495	0.414	0.367	0.334
β	1.703	1.721	1.728	1.732	1.732	1.707	1.671	1.680	1.700

sections have been made by S. Timoshenko (1923). An experimental investigation by D. C. Broughton, M. E. Clark, and H. T. Corten (1950) showed that another type of correction is needed if the curved beam has extremely thick flanges and thin webs. For such beams each flange tends to rotate about a neutral axis of its own in addition to the rotation about the neutral axis of the curved beam cross section as a whole. Curved beams for which the circumferential stresses are appreciably increased by this action probably fail by excessive radial stresses.

Note: The radial stress can be calculated using either the original or the modified cross section.

EXAMPLE 9.8
Bleich Correction Factors for T-Section

A T-section curved beam has the dimensions indicated in Figure E9.8*a* and is subjected to pure bending. The curved beam is made of a steel having a yield stress Y = 280 MPa.

(a) Determine the magnitude of the moment that indicates yielding in the curved beam if Bleich's correction factors are not used.

(b) Use Bleich's correction factors to obtain a modified cross section. Determine the magnitude of the moment that initiates yielding for the modified cross section and compare with the result of part (a).

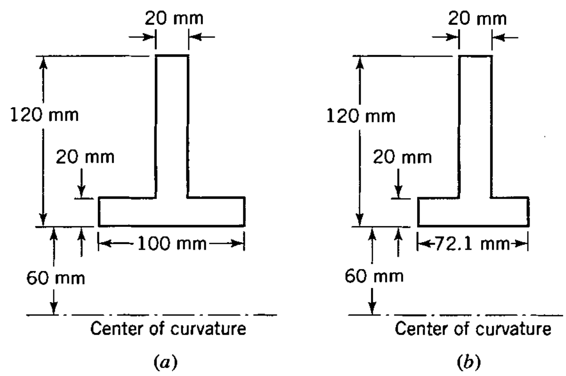


FIGURE E9.8 (*a*) Actual section. (*b*) Modified section.

Solution

(a) The magnitudes of A, A_m, and R for the actual cross section are given by Eqs. 9.12, 9.13, and 9.14, respectively, as follows: A = 4000 mm^2, A_m = 44.99 mm, and R = 100.0 mm. By comparison of the stresses at the locations r = 180 mm and r = 60 mm, we find that the maximum magnitude of $\sigma_{\theta\theta}$ occurs at the outer radius (r = 180 mm). (See Eq. 9.11.) Thus,

$$\sigma_{\theta\theta(\max)} = \left|\frac{M_x[4000 - 180(44.99)]}{4000(180)[100.0(44.9) - 4000]}\right|$$
$$= \left|-1.141 \times 10^{-5} M_x\right|$$

where M_x has the units of N • mm. Since the state of stress is assumed to be uniaxial, the magnitude of M_x to initiate yielding is obtained by setting $\sigma_{\theta\theta} = -Y$. Thus,

$$M_x = \frac{280}{1.141 \times 10^{-5}} = 24{,}540{,}000 \text{ N} \bullet \text{mm} = 24.54 \text{ kN} \bullet \text{m}$$

(b) The dimensions of the modified cross section are computed by Bleich's method; hence $b_p^2/\bar{r}t_f$ must be calculated. It is

$$\frac{b_p^2}{\bar{r}t_f} = \frac{40(40)}{70(20)} = 1.143$$

Linear interpolation in Table 9.3 yields $\alpha = 0.651$ and $\beta = 1.711$. Hence, by Eqs. 9.22 and 9.23, the modified flange width is $b_p' = \alpha b_p = 0.651(40) = 26.04$ mm and $b' = 2b_p' + t_w = 2(26.04) + 20 = 72.1$ mm (Figure E9.8*b*). For this cross section, by means of Eqs. 9.12, 9.13, and 9.14, we find

$$A = 72.1(20) + 20(100) = 3442 \text{ mm}^2$$

$$R = \frac{72.1(20)(70) + 20(100)(130)}{3442} = 104.9 \text{ mm}$$

$$A_m = 72.1 \ln\frac{80}{60} + 20 \ln\frac{180}{80} = 36.96 \text{ mm}$$

Now by means of Eq. 9.11, we find that the maximum magnitude of $\sigma_{\theta\theta}$ occurs at the inner radius of the modified cross section. Thus, with $r = 60$ mm, Eq. 9.11 yields

$$\sigma_{\theta\theta(\max)} = \frac{M_x[3442 - 60(36.96)]}{3442(60)[104.9(36.96) - 3442]} = 1.363 \times 10^{-5} M_x$$

The magnitude of M_x that causes yielding can be calculated by means of either the maximum shear stress criterion of failure or the octahedral shear stress criterion of failure. If the maximum shear stress criterion is used, the minimum principal stress must also be computed. The minimum principal stress is σ_{xx}. Hence, by Eqs. 9.11 and 9.24, we find

$$\bar{\sigma}_{\theta\theta} = \frac{M_x[3442 - 70(36.96)]}{3442(70)[104.9(36.96) - 3442]} = 8.15 \times 10^{-6} M_x$$

$$\sigma_{xx} = -\beta\bar{\sigma}_{\theta\theta} = -1.711(8.15 \times 10^{-6} M_x) = -1.394 \times 10^{-5} M_x$$

and

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{Y}{2} = \frac{\sigma_{\theta\theta(\max)} - \sigma_{xx}}{2}$$

$$M_x = 10{,}140{,}000 \text{ N} \cdot \text{mm} = 10.14 \text{ kN} \cdot \text{m}$$

A comparison of the moment M_x determined in parts (a) and (b) indicates that the computed M_x required to initiate yielding is reduced by 58.8% because of the distortion of the cross section. Since the yielding is highly localized, its effect is not of concern unless the curved beam is subjected to fatigue loading. If the second principal stress σ_{xx} is neglected, the moment M_x is reduced by 16.5% because of the distortion of the cross section. The distortion is reduced if the flange thickness is increased.

DEFLECTIONS OF CURVED BEAMS

A convenient method for determining the deflections of a linearly elastic curved beam is by the use of Castigliano's theorem (Chapter 5). For example, the deflection and rotation of the free end of the curved beam in Figure 9.2*a* are given by the relations

$$\delta_{P_1} = \frac{\partial U}{\partial P_1} \tag{9.25}$$

$$\phi = \frac{\partial U}{\partial M_0} \tag{9.26}$$

where δ_{P_1} is the component of the deflection of the free end of the curved beam in the direction of load P_1, ϕ is the angle of rotation of the free end of the curved beam in the direction of M_0, and U is the total elastic strain energy in the curved beam. The total strain energy U (see Eq. 5.6) is equal to the integral of the strain-energy density U_0 over the volume of the curved beam (see Eqs. 3.33 and 5.7).

Consider the strain-energy density U_0 for a curved beam (Figure 9.2). Because of the symmetry of loading relative to the (y, z) plane, $\sigma_{xy} = \sigma_{xz} = 0$, and since the effect of the transverse normal stress σ_{xx} (Figure 9.2b) is ordinarily neglected, the formula for the strain-energy density U_0 reduces to the form

$$U_0 = \frac{1}{2E}\sigma_{\theta\theta}^2 + \frac{1}{2E}\sigma_{rr}^2 - \frac{\nu}{E}\sigma_{rr}\sigma_{\theta\theta} + \frac{1}{2G}\sigma_{r\theta}^2$$

where the radial normal stress σ_{rr}, the circumferential normal stress $\sigma_{\theta\theta}$, and the shear stress $\sigma_{r\theta}$ are, relative to the (x, y, z) axes of Figure 9.2b, $\sigma_{rr} = \sigma_{yy}$, $\sigma_{\theta\theta} = \sigma_{zz}$, and $\sigma_{r\theta} = \sigma_{yz}$. In addition, the effect of σ_{rr} is often small for curved beams of practical dimensions. Hence, the effect of σ_{rr} is often discarded from the expression for U_0. Then,

$$U_0 = \frac{1}{2E}\sigma_{\theta\theta}^2 + \frac{1}{2G}\sigma_{r\theta}^2$$

The stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, respectively, contribute to the strain energies U_N and U_S because of the normal traction N and shear V (Figure 9.2b). In addition, $\sigma_{\theta\theta}$ contributes to the bending strain energy U_M, as well as to the strain energy U_{MN} because of a coupling effect between the moment M and traction N, as we shall see in the derivation below.

Ordinarily, it is sufficiently accurate to approximate the strain energies U_S and U_N that are due to shear V and traction N, respectively, by the formulas for straight beams (see Section 5.3). However, the strain energy U_M resulting from bending must be modified. To compute this strain energy, consider the curved beam shown in Figure 9.2b. Since the strain energy increment dU for a linearly elastic material undergoing small displacement is independent of the order in which loads are applied, let the shear load V and normal load N be applied first. Next, let the moment be increased from zero to M_x. The strain energy increment resulting from bending is

$$dU_M = \frac{1}{2}M_x\Delta(d\theta) = \frac{1}{2}M_x\omega\, d\theta \tag{9.27}$$

where $\Delta(d\theta)$, the change in $d\theta$, and $\omega = \Delta(d\theta)/d\theta$ are due to M_x alone. Hence, ω is determined from Eq. 9.10 with $N = 0$. Consequently, Eqs. 9.27 and Eq. 9.10 yield (with $N = 0$)

$$dU_M = \frac{A_m M_x^2}{2A(RA_m - A)E}\, d\theta \tag{9.28}$$

During the application of M_x, additional work is done by N because the centroidal (middle) surface (Figure 9.2b) is stretched an amount $d\bar{e}_{\theta\theta}$. Let the corresponding strain energy increment caused by the stretching of the middle surface be denoted by dU_{MN}. This strain energy increment dU_{MN} is equal to the work done by N as it moves through the distance $d\bar{e}_{\theta\theta}$. Thus,

$$dU_{\mathrm{MN}} = Nd\bar{e}_{\theta\theta} = N\bar{\epsilon}_{\theta\theta}R\,d\theta \tag{9.29}$$

where $d\bar{e}_{\theta\theta}$ and $\bar{e}_{\theta\theta}$ refer to the elongation and strain of the centroidal axis, respectively. The strain $\bar{e}_{\theta\theta}$ is given by Eq. 9.3 with $r = R$. Thus, Eq. 9.3 (with $r = R$) and Eqs. 9.29, 9.9, and 9.10 (with $N = 0$) yield the strain energy increment dU_{MN} resulting from coupling of the moment M_x and traction N:

$$dU_{\mathrm{MN}} = \frac{N}{E}\left[\frac{M_x}{RA_m - A} - R\frac{A_m M_x}{A(RA_m - A)}\right]d\theta = -\frac{M_x N}{EA}d\theta \tag{9.30}$$

By Eqs. 5.8, 5.14, 9.28, and 9.30, the total strain energy U for the curved beam is obtained in the form

$$U = U_{\mathrm{S}} + U_{\mathrm{N}} + U_{\mathrm{M}} + U_{\mathrm{MN}}$$

or

$$U = \int\frac{kV^2R}{2AG}d\theta + \int\frac{N^2R}{2AE}d\theta + \int\frac{A_m M_x^2}{2A(RA_m - A)E}d\theta - \int\frac{M_x N}{EA}d\theta \tag{9.31}$$

Equation 9.31 is an approximation, since it is based on the assumptions that plane sections remain plane and that the effect of the radial stress σ_{rr} on U is negligible. It might be expected that the radial stress increases the strain energy. Hence, Eq. 9.31 yields a low estimate of the actual strain energy. However, if M_x and N have the same sign, the coupling U_{MN}, the last term in Eq. 9.31, is negative. Ordinarily, U_{MN} is small and, in many cases, it is negative. Hence, we recommend that U_{MN}, the coupling strain energy, be discarded from Eq. 9.31 when it is negative. The discarding of U_{MN} from Eq. 9.31 raises the estimate of the actual strain energy when U_{MN} is negative and compensates to some degree for the lower estimate caused by discarding σ_{rr}.

The deflection δ_{elast} of rectangular cross section curved beams has been given by Timoshenko and Goodier (1970) for the two types of loading shown in Figure 9.4. The ratio of the deflection δ_U given by Castigliano's theorem and the deflection δ_{elast} is presented in Table 9.4 for several values of R/h. The shear coefficient k (see Eqs. 5.14 and 5.15) was taken to be 1.5 for the rectangular section, and Poisson's ratio v was assumed to be 0.30.

TABLE 9.4 Ratios of Deflections in Rectangular Section Curved Beams Computed by Elasticity Theory and by Approximate Strain Energy Solution

	Neglecting U_{MN}		Including U_{MN}	
	Pure bending	Shear loading	Pure bending	Shear loading
$\left(\frac{R}{h}\right)$	$\left(\frac{\delta_U}{\delta_{\mathrm{elast}}}\right)$	$\left(\frac{\delta_U}{\delta_{\mathrm{elast}}}\right)$	$\left(\frac{\delta_U}{\delta_{\mathrm{elast}}}\right)$	$\left(\frac{\delta_U}{\delta_{\mathrm{elast}}}\right)$
0.65	0.923	1.563	0.697	1.215
0.75	0.974	1.381	0.807	1.123
1.0	1.004	1.197	0.914	1.048
1.5	1.006	1.085	0.968	1.016
2.0	1.004	1.048	0.983	1.008
3.0	1.002	1.021	0.993	1.003
5.0	1.000	1.007	0.997	1.001

Note: The deflection of curved beams is much less influenced by the curvature of the curved beam than is the circumferential stress $\sigma_{\theta\theta}$. If R/h is greater than 2.0, the strain energy resulting from bending can be approximated by that for a straight beam. Thus, for $R/h > 2.0$, for computing deflections the third and fourth terms on the right-hand side of Eq. 9.31 may be replaced by

$$U_M = \int \frac{M_x^2}{2EI_x} R\, d\theta \tag{9.32}$$

In particular, we note that the deflection of a rectangular cross section curved beam with $R/h = 2.0$ is 7.7% greater when the curved beam is assumed to be straight than when it is assumed to be curved.

9.5.1 Cross Sections in the Form of an I, T, etc.

As discussed in Section 9.4, the cross sections of curved beams in the form of an I, T, etc. undergo distortion when loaded. One effect of the distortion is to decrease the stiffness of the curved beam. As a result, deflections calculated on the basis of the undistorted cross section are less than the actual deflections. Therefore, the deflection calculations should be based on modified cross sections determined by Bleich's correction factors (Table 9.3). The strain energy terms U_N and U_M for the curved beams should also be calculated using the modified cross section. We recommend that the strain energy U_S be calculated with $k = 1.0$, and with the cross-sectional area A replaced by the area of the web $A_w = th$, where t is the thickness of the web and h is the curved beam depth. Also, as a working rule, we recommend that the coupling energy U_{MN} be neglected if it is negative and that it be doubled if it is positive.

EXAMPLE 9.9
Deformations in a Curved Beam Subjected to Pure Bending

The curved beam in Figure E9.9 is made of an aluminum alloy (E = 72.0 GPa), has a rectangular cross section with a thickness of 60 mm, and is subjected to a pure bending moment M = 24.0 kN • m.

(a) Determine the angle change between the two horizontal faces where M is applied.

(b) Determine the relative displacement of the centroids of the horizontal faces of the curved beam.

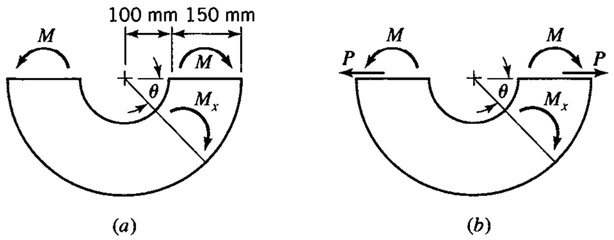


FIGURE E9.9

Solution

Required values for A, A_m, and R for the curved beam are calculated using equations in row (a) of Table 9.2:

$$A = 60(150) = 9000 \text{ mm}^2$$
$$A_m = 60 \ln\frac{250}{100} = 54.98 \text{ mm}$$
$$R = 100 + 75 = 175 \text{ mm}$$

(a) The angle change between the two faces where M is applied is given by Eq. 9.26. As indicated in Figure E9.9*a*, the magnitude of M_x at any angle θ is $M_x = M$. Thus, by Eq. 9.26, we obtain

$$\phi = \frac{\partial U}{\partial M} = \int_0^{\pi} \frac{A_m M_x}{A(RA_m - A)E}(1)\, d\theta$$

$$= \frac{54.98(24{,}000{,}000)\pi}{9000[175(54.98) - 9000](72{,}000)}$$

$$= 0.01029 \text{ rad}$$

(b) To determine the deflection of the curved beam, a load P must be applied as indicated in Figure E9.9*b*. In this case, $M_x = M + PR\sin\theta$ and $\partial U/\partial P = R\sin\theta$. Then the deflection is given by Eq. 9.25, in which the integral is evaluated with $P = 0$. Thus, the relative displacement is given by the relation

$$\delta_P = \frac{\partial U}{\partial P} = \int_0^{\pi} \frac{A_m M_x}{A(RA_m - A)E}\bigg|_{P=0} (R\sin\theta)\, d\theta$$

or

$$\delta_P = \frac{54.98(24{,}000{,}000)(175)(2)}{9000[175(54.98) - 9000](72{,}000)} = 1.147 \text{ mm}$$

EXAMPLE 9.10 *Deflections in a Press*

A press (Figure E9.10*a*) has the cross section shown in Figure E9.10*b*. It is subjected to a load P = 11.2 kN. The press is made of steel with E = 200 GPa and ν = 0.30. Determine the separation of the jaws of the press caused by the load.

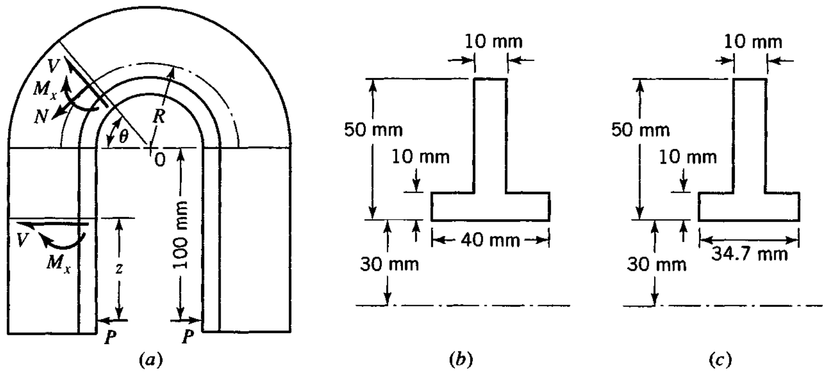


FIGURE E9.10 (*a*) Curved beam. (*b*) Actual section. (c) Modified section.

Solution

The press is made up of two straight members and a curved member. We compute the strain energies resulting from bending and shear in the straight beams, without modification of the cross sections. The moment of inertia of the cross section is $I_x = 181.7 \times 10^3$ mm^4. We choose the origin of the coordinate axes at load P, with z measured from P toward the curved beam. Then the applied shear V and moment M_x at a section in the straight beam are

$$V = P$$
$$M_x = Pz$$

In the curved beam portion of the press, we employ Bleich's correction factor to obtain a modified cross section. With the dimensions in Figure E9.10*b*, we find

$$\frac{b_p^2}{\bar{r}t_f} = \frac{15^2}{35(10)} = 0.643$$

A linear interpolation in Table 9.3 yields the result $\alpha = 0.822$. The modified cross section is shown in Figure E9.10*c*. Equations 9.12–9.14 give

$$A = 34.7(10) + 10(40) = 747 \text{ mm}^2$$

$$R = \frac{34.7(10)(35) + 10(40)(60)}{747} = 48.4 \text{ mm}$$

$$A_m = 10 \ln\frac{80}{40} + 34.7 \ln\frac{40}{30} = 16.9 \text{ mm}$$

With θ defined as indicated in Figure E9.10*a*, the applied shear *V*, normal load *N*, and moment M_x for the curved beam are

$$V = P\cos\theta$$
$$N = P\sin\theta$$
$$M_x = P(100 + R\sin\theta)$$

Summing the strain energy terms for the two straight beams and the curved beam and taking the derivative with respect to *P* (Eq. 9.25), we compute the increase in distance δ_P between the load points as

$$\delta_P = 2\int_0^{100} \frac{P}{A_w G}\,dz + 2\int_0^{100} \frac{Pz^2}{EI_x}\,dz + \int_0^{\pi} \frac{P\cos^2\theta}{A_w G} R\,d\theta + \int_0^{\pi} \frac{P\sin^2\theta}{AE} R\,d\theta + \int_0^{\pi} \frac{P(100 + R\sin\theta)^2 A_m}{A(RA_m - A)E}\,d\theta$$

The shear modulus is $G = E/[2(1+\nu)] = 76{,}900$ MPa and $A_w = th = (10)(50) = 500 \text{ mm}^2$. Hence,

$$\delta_P = \frac{2(11{,}200)(100)}{76{,}900(500)} + \frac{2(11{,}200)(100)^3}{3(200{,}000)(181{,}700)}$$
$$+ \frac{11{,}200(48.4)\pi}{500(76{,}900)(2)} + \frac{11{,}200(48.4)\pi}{747(200{,}000)(2)}$$
$$+ \frac{16.9(11{,}200)}{747[48.4(16.9) - 747](200{,}000)}\left[(100)^2\pi + \frac{\pi}{2}(48.4)^2 + 2(100)(48.4)(2)\right]$$

or

$$\delta_P = 0.058 + 0.205 + 0.022 + 0.006 + 0.972 = 1.263 \text{ mm}$$

STATICALLY INDETERMINATE CURVED BEAMS: CLOSED RING SUBJECTED TO A CONCENTRATED LOAD

Many curved members, such as closed rings and chain links, are statically indeterminate (see Section 5.5). For such members, equations of equilibrium are not sufficient to determine all the internal resultants (V, N, M_x) at a section of the member. The additional relations needed to solve for the loads are obtained using Castigliano's theorem with appropriate boundary conditions. Since closed rings are commonly used in engineering, we present the computational procedure for a closed ring.

Consider a closed ring subjected to a central load P (Figure 9.9*a*). From the condition of symmetry, the deformations of each quadrant of the ring are identical. Hence, we need consider only one quadrant. The quadrant (Figure 9.9*b*) may be considered fixed at section FH with a load $P/2$ and moment M_0 at section BC. Because of the symmetry of the ring, as the ring deforms, section BC remains perpendicular to section FH. Therefore, by Castigliano's theorem, we have for the rotation of face BC

$$\phi_{BC} = \frac{\partial U}{\partial M_0} = 0 \tag{9.33}$$

The applied loads V, N, and M_x at a section forming angle θ with the face BC are

$$\begin{aligned} V &= \frac{P}{2}\sin\theta \\ N &= \frac{P}{2}\cos\theta \\ M_x &= M_0 - \frac{PR}{2}(1-\cos\theta) \end{aligned} \tag{9.34}$$

Substituting Eqs. 9.31 and 9.34 into Eq. 9.33, we find

$$0 = \int_0^{\pi/2} \frac{\left[M_0 - \left(\frac{PR}{2}\right)(1-\cos\theta)\right]A_m}{A(RA_m - A)E}\, d\theta - \int_0^{\pi/2} \frac{\left(\frac{P}{2}\right)\cos\theta}{AE}\, d\theta \tag{9.35}$$

where U_{MN} has been included. The solution of Eq. 9.35 is

$$M_0 = \frac{PR}{2}\left(1 - \frac{2A}{RA_m\pi}\right) \tag{9.36}$$

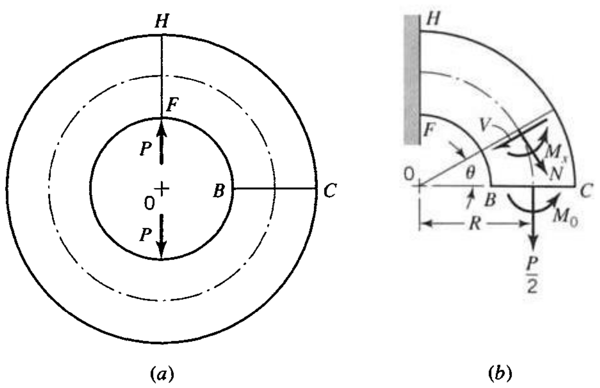


FIGURE 9.9 Closed ring.

If R/h is greater than 2.0, we take the bending energy U_M as given by Eq. 9.32 and ignore the coupling energy U_{MN}. Then, M_0 is given by the relation

$$M_0 = \frac{PR}{2}\left(1 - \frac{2}{\pi}\right) \tag{9.37}$$

With M_0 known, the loads at every section of the closed ring (Eqs. 9.34) are known. The stresses and deformations of the closed ring may be calculated by the methods of Sections 9.2–9.5.

9.7 FULLY PLASTIC LOADS FOR CURVED BEAMS

In this section we consider curved beams made of elastic–perfectly plastic materials with yield stress Y (Figure 1.5*b*). For a curved beam made of elastic–perfectly plastic material, the fully plastic moment M_P under pure bending is the same as that for a straight beam with identical cross section and material. However, because of the nonlinear distribution of the circumferential stress $\sigma_{\theta\theta}$ in a curved beam, the ratio of the fully plastic moment M_P under pure bending to maximum elastic moment M_Y is much greater for a curved beam than for a straight beam with the same cross section.

Most curved beams are subjected to complex loading other than pure bending. The stress distribution for a curved beam at the fully plastic load P_P for a typical loading condition is indicated in Figure 9.10. Since the tension stresses must balance the compression stresses and load P_P, the part A_T of the cross-sectional area A that has yielded in tension is larger than the part A_C of area A that has yielded in compression. In addition to the unknowns A_T and A_C, a third unknown is P_P, the load at the fully plastic condition. This follows because R (the distance from the center of curvature 0 to the centroid $\bar{0}$) can be calculated and D is generally specified rather than P_P. The three equations necessary to determine the three unknowns A_T, A_C, and P_P are obtained from the equations of equilibrium and the fact that the sum of A_T and A_C must equal the cross-sectional area A, that is,

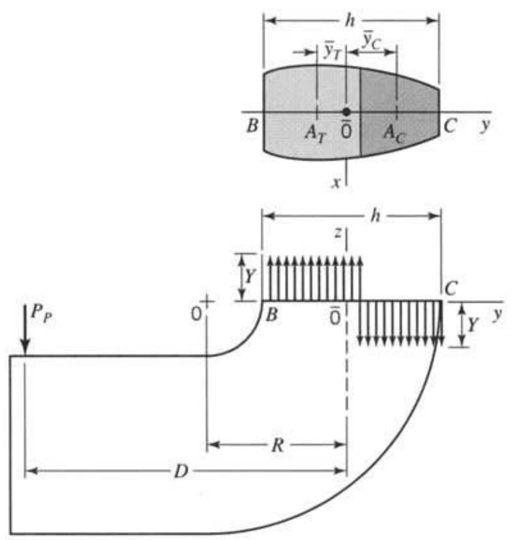


FIGURE 9.10 Stress distribution for a fully plastic load on a curved beam.

$$A = A_T + A_C \tag{9.38}$$

The equilibrium equations are (Figure 9.10)

$$\sum F_z = 0 = A_T Y - A_C Y - P_P \tag{9.39}$$

$$\sum M_x = 0 = P_P D - A_T Y \bar{y}_T - A_C Y \bar{y}_C \tag{9.40}$$

In Eq. 9.40, $\bar{y}_T$ and $\bar{y}_C$ locate the centroids of A_T and A_C, respectively, as measured from the centroid $\bar{0}$ of the cross-sectional area of the curved beam (Figure 9.10). Let M be the moment, about the centroidal axis x, resulting from the stress distribution on section BC (Figure 9.10). Then,

$$M = P_P D = A_T Y \bar{y}_T + A_C Y \bar{y}_C \tag{9.41}$$

Trial and error can be used to solve Eqs. 9.38–9.40 for the magnitudes of A_T, A_C, and P_P, since $\bar{y}_T$ and $\bar{y}_C$ are not known until A_T and A_C are known (McWhorter et al., 1971).

The moment M (Eq. 9.41) is generally less than the fully plastic moment M_P for pure bending. It is desirable to know the conditions under which M resulting from load P_P can be assumed equal to M_P, since for pure bending A_T is equal to A_C, and the calculations are greatly simplified. For some common sections, $M \approx M_P$, when $D > h$. For example, for $D = h$, we note that $M = 0.94M_P$ for curved beams with rectangular sections and $M = 0.96M_P$ for curved beams with circular sections. However, for curved beams with T-sections, M may be greater than M_P. Other exceptions are curved beams with I-sections and box-sections, for which D should be greater than $2h$ for M to be approximately equal to M_P.

9.7.1 Fully Plastic Versus Maximum Elastic Loads for Curved Beams

A linearly elastic analysis of a load-carrying member is required to predict the load–deflection relation for linearly elastic behavior of the member up to the load P_Y that initiates yielding in the member. The fully plastic load is also of interest since it is often considered to be the limiting load that can be applied to the member before the deformations become excessively large.

The fully plastic load P_P for a curved beam is often more than twice the maximum elastic load P_Y. Fracture loads for curved beams that are made of ductile metals and subjected to static loading may be four to six times P_Y. Dimensionless load–deflection experimental data for a uniform rectangular section hook made of a structural steel are shown in Figure 9.11. The deflection is defined as the change in distance ST between points S and T on the hook. The hook does not fracture even for loads such that $P/P_Y > 5$. A computer program written by J. C. McWhorter, H. R. Wetenkamp, and O. M. Sidebottom (1971) gave the predicted curve in Figure 9.11. The experimental data agree well with predicted results.

As noted in Figure 9.11, the ratio of P_P to P_Y is 2.44. Furthermore, the load–deflection curve does not level off at the fully plastic load but continues to rise. This behavior may be attributed to strain hardening. Because of the steep stress gradient in the hook, the strains in the most strained fibers become so large that the material begins to strain harden before yielding can penetrate to sufficient depth at section BC in the hook to develop the fully plastic load.

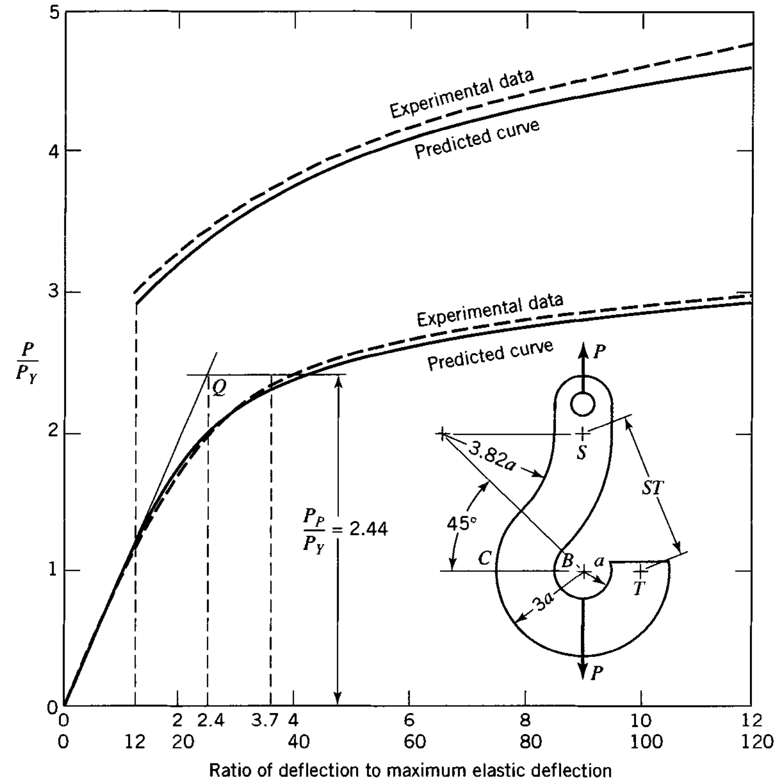


FIGURE 9.11 Dimensionless load–deflection curves for a uniform rectangular section hook made of structural steel.

The usual practice in predicting the deflection of a structure at the fully plastic load is to assume that the structure behaves in a linearly elastic manner up to the fully plastic load (point Q in Figure 9.11) and multiply the deflection at this point by the ratio P_P/P_Y (in this case, 2.44). In this case, with this procedure (Figure 9.11) the resulting calculated deflection [approximately calculated as 2.44(2.4) = 5.9] is greater than the measured deflection.

Usually, curved members such as crane hooks and chains are not subjected to a sufficient number of repetitions of peak loads during their life for fatigue failure to occur. Therefore, the working loads for these members are often obtained by application of a factor of safety to the fully plastic loads. It is not uncommon to have the working load as great as or greater than the maximum elastic load P_Y.

PROBLEMS

Section 9.2

9.1. A curved beam has the T-shaped cross section shown in Figure P9.1. The radius of curvature to the inner face of the flange is 20 mm. The maximum allowable circumferential stress has a magnitude of 250 MPa. Determine the magnitude of the bending moment that may be applied to the beam.

9.2. A curved steel bar of circular cross section is used as a crane hook (Figure P9.2). The radius of curvature to the inner edge of the bar is r and the bar has diameter d.

a. Determine the maximum tensile and compressive stresses at section A–A in terms of load P, radius r, and diameter d.

b. The maximum allowable design tensile stress at section A–A is 375 MPa. Determine the maximum allowable load P, for a radius r = 75 mm and a diameter d = 50 mm.

9.3. In a redesign of the aircraft beam of Example 9.2, the beam is replaced by a beam with the cross section shown in Figure P9.3.

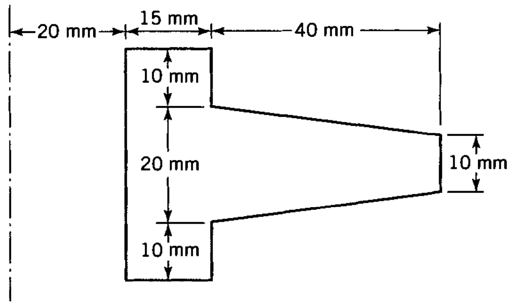


FIGURE P9.1

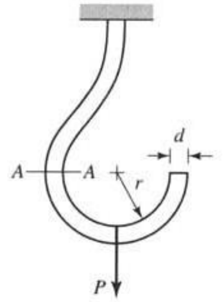


FIGURE P9.2

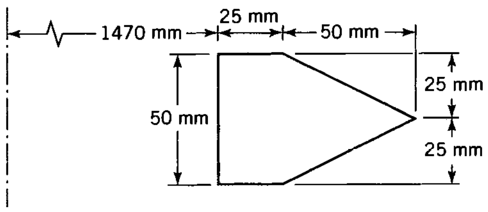


FIGURE P9.3

a. Rework Example 9.2 with the new cross section.

b. Compare the results to those of Example 9.2.

c. Comment on the worthiness of the redesign.

9.4. Rework Example 9.4 assuming that the pin exerts a uniform pressure p on the hook at radius r_i for $0 \le \theta \le \pi$. Compare the results to those of Example 9.4.

9.5. The frame shown in Figure E9.1 has a rectangular cross section with a thickness of 10 mm and depth of 40 mm. The load P is located 120 mm from the centroid of section BC. The frame is made of steel having a yield stress of Y = 430 MPa. The frame has been designed using a factor of safety of SF = 1.75 against initiation of yielding. Determine the maximum allowable magnitude of P, if the radius of curvature at section BC is R = 40 mm.

9.6. Solve Problem 9.5 for the condition that R = 35 mm.

9.7. The curved beam in Figure P9.7 has a circular cross section 50 mm in diameter. The inside diameter of the curved beam is 40 mm. Determine the stress at B for P = 20 kN.

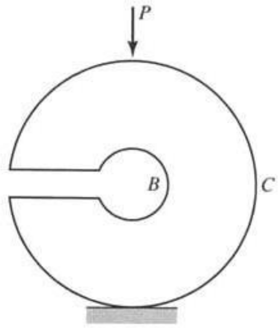


FIGURE P9.7

9.8. Let the crane hook in Figure E9.3 have a trapezoidal cross section as shown in row (*c*) of Table 9.2 with (see Figure P9.8) a = 45 mm, c = 80 mm, b_1 = 25 mm, and b_2 = 10 mm. Determine the maximum load to be carried by the hook if the working stress limit is 150 MPa.

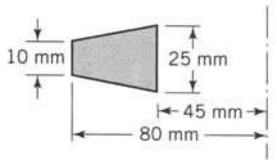


FIGURE P9.8

9.9. A curved beam is built up by welding together rectangular and elliptical cross section curved beams; the cross section is shown in Figure P9.9. The center of curvature is located 20 mm from B. The curved beam is subjected to a positive bending moment M_x. Determine the stresses at points B and C in terms of M_x.

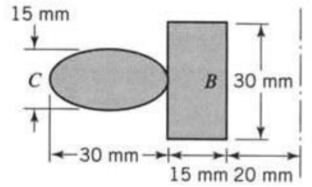


FIGURE P9.9

9.10. A commercial crane hook has the cross-sectional dimensions shown in Figure P9.10 at the critical section that is subjected to an axial load P = 100 kN. Determine the circumferential stresses at the inner and outer radii for this load.

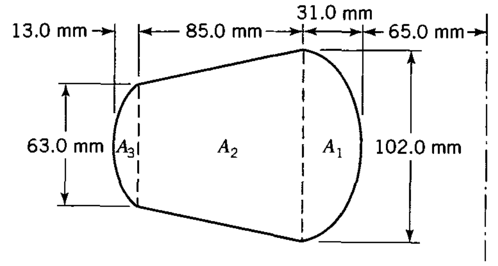


FIGURE P9.10

Assume that area A_1 is half of an ellipse [see row (j) in Table 9.2] and area A_3 is enclosed by a circular arc.

9.11. A crane hook has the cross-sectional dimensions shown in Figure P9.11 at the critical section that is subjected to an axial load P = 90.0 kN. Determine the circumferential stresses at the inner and outer radii for this load. Note that A_1 and A_3 are enclosed by circular arcs.

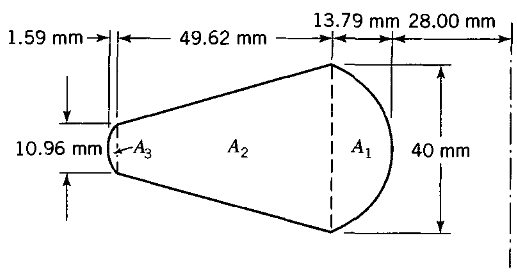


FIGURE P9.11

9.12. The curved beam in Figure P9.12 has a triangular cross section with the dimensions shown. If P = 40 kN, determine the circumferential stresses at B and C.

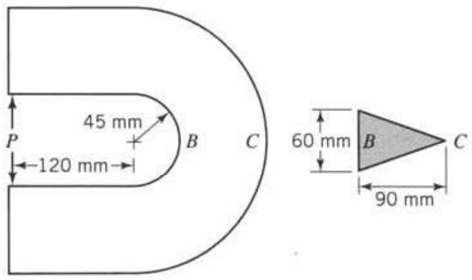


FIGURE P9.12

9.13. A curved beam with a rectangular cross section strikes a 90° arc and is loaded and supported as shown in Figure P9.13. The thickness of the beam is 50 mm. Determine the hoop stress $\sigma_{\theta\theta}$ along line A–A at the inside and outside radii and at the centroid of the beam.

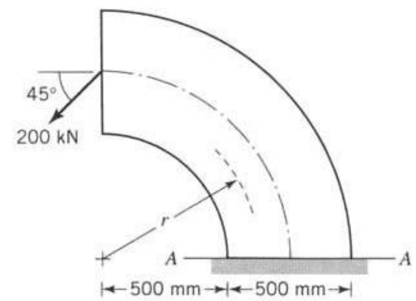


FIGURE P9.13

Section 9.3

9.14. Determine the distribution of the radial stress σ_{rr} in section BC of the beam of Example 9.1. Also determine the maximum value of σ_{rr} and its location.

9.15. Determine the magnitude of the radial stress σ_{rr} in section BC of Figure P9.12 at a radial distance of 30 mm from point B.

9.16. For the curved beam in Problem 9.9, determine the radial stress in terms of the moment M_x if the thickness of the web at the weld is 10 mm.

9.17. Figure P9.17 shows a cast iron frame with a U-shaped cross section. The ultimate tensile strength of the cast iron is σ_u = 320 MPa.

a. Determine the maximum value of P based on a factor of safety SF = 4.00, which is based on the ultimate strength.

b. Neglecting the effect of stress concentrations at the fillet at the junction of the web and flange, determine the maximum radial stress when this load is applied.

c. Is the maximum radial stress less than the maximum circumferential stress?

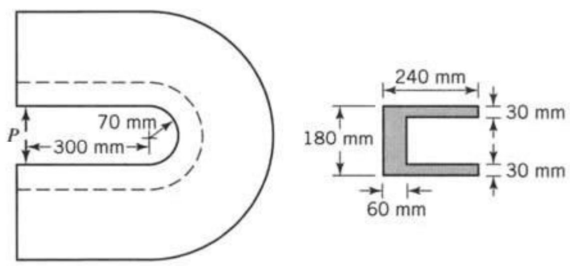


FIGURE P9.17

Section 9.4

9.18. A T-section curved beam has the cross section shown in Figure P9.18. The center of curvature lies 40 mm from the flange. If the curved beam is subjected to a positive bending moment $M_x = 2.50$ kN • m, determine the stresses at the inner and outer radii. Use Bleich's correction factors. What is the maximum shear stress in the curved beam?

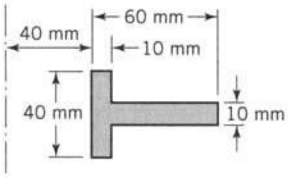


FIGURE P9.18

9.19. Determine the radial stress at the junction of the web and the flange for the curved beam in Problem 9.18. Neglect stress concentrations. Use the Bleich correction.

9.20. A load $P = 12.0$ kN is applied to the clamp shown in Figure P9.20. Determine the circumferential stresses at points B and C, assuming that the curved beam formula is valid at that section.

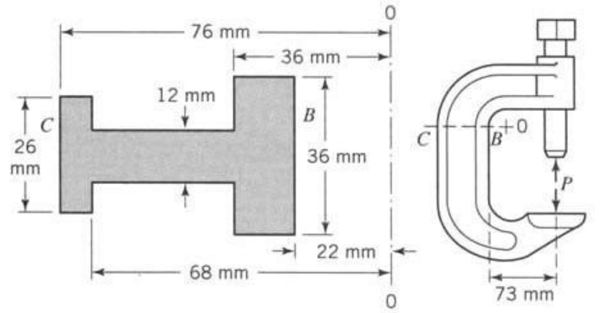


FIGURE P9.20

9.21. Determine the radial stress at the junction of the web and inner flange of the curved beam portion of the clamp in Problem 9.20. Neglect stress concentrations.

Section 9.5

9.22. If moment M_x and axial force N are applied simultaneously, the strain-energy density resulting from these two actions is

$$dU = \frac{1}{2}M_x\omega d\theta + \frac{1}{2}N\bar{\epsilon}_{\theta\theta}R\,d\theta$$

where ω is given by Eq. 9.10 and $\bar{\epsilon}_{\theta\theta}$ is found from Eq. 9.3 with $r = R$. Using this expression for strain-energy density, derive Eq. 9.31.

9.23. The curved beam in Figure P9.23 is made of a steel (E = 200 GPa) that has a yield stress $Y = 420$ MPa. Determine the magnitude of the bending moment M_Y required to initiate yielding in the curved beam, the angle change of the free end, and the horizontal and vertical components of the deflection of the free end.

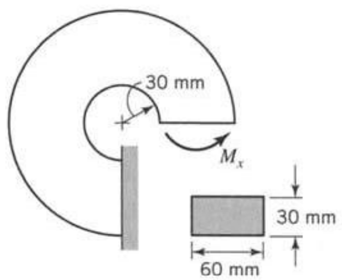


FIGURE P9.23

9.24. Determine the deflection of the curved beam in Problem 9.7 at the point of load application. The curved beam is made of an aluminum alloy for which $E = 72.0$ GPa and $G = 27.1$ GPa. Let $k = 1.3$.

9.25. The triangular cross section curved beam in Problem 9.12 is made of steel (E = 200 GPa and G = 77.5 GPa). Determine the separation of the points of application of the load. Let $k = 1.5$.

9.26. Determine the deflection across the center of curvature of the cast iron curved beam in Problem 9.17 for P = 126 kN. E = 102.0 GPa and $G = 42.5$ GPa. Let $k = 1.0$ with the area in shear equal to the product of the web thickness and the depth.

Section 9.6

9.27. The ring in Figure P9.27 has an inside diameter of 100 mm, an outside diameter of 180 mm, and a circular cross section. The ring is made of steel having a yield stress of Y = 520 MPa. Determine the maximum allowable magnitude of P if the ring has been designed with a factor of safety SF = 1.75 against initiation of yielding.

9.28. If E = 200 GPa and G = 77.5 GPa for the steel in Problem 9.27, determine the deflection of the ring for a load P = 60 kN. Let $k = 1.3$.

9.29. An aluminum alloy ring has a mean diameter of 600 mm and a rectangular cross section with 200 mm thickness and a depth of 300 mm (radial direction). The ring is loaded by

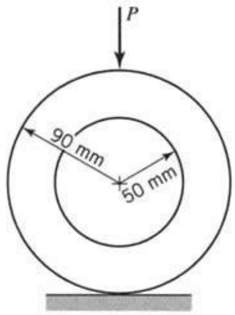


FIGURE P9.27

diametrically opposed radial loads $P = 4.00$ MN. Determine the maximum tensile and compressive circumferential stresses in the ring.

9.30. If $E = 72.0$ GPa and $G = 27.1$ GPa for the aluminum alloy ring in Problem 9.29, determine the separation of the points of application of the loads. Let $k = 1.5$.

9.31. The link in Figure P9.31 has a circular cross section and is made of a steel having a yield stress of $Y = 250$ MPa. Determine the magnitude of P that will initiate yield in the link.

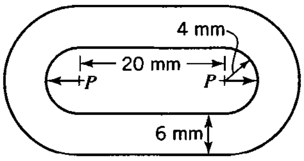


FIGURE P9.31

Section 9.7

9.32. Let the curved beam in Figure 9.10 have a rectangular cross section with depth h and width b. Show that the ratio of the bending moment M for fully plastic load P_P to the fully plastic moment for pure bending $M_P = Ybh^2/4$ is given by the relation

$$\frac{M}{M_P} = \frac{4D}{h}\sqrt{1 + \frac{4D^2}{h^2} - \frac{8D^2}{h^2}}$$

9.33. Let the curved beam in Problem 9.5 be made of a steel that has a flat-top stress–strain diagram at the yield stress $Y =$ 430 MPa. From the answer to Problem 9.5, the load that initiates yielding is equal to $P_Y = SF(P) = 6.05$ kN. Since $D = 3h$, assume $M = M_P$ and calculate P_P. Determine the ratio P_P/P_Y.

9.34. Let the steel in the curved beam in Example 9.8 be elastic–perfectly plastic with yield stress $Y = 280$ MPa. Determine the fully plastic moment for the curved beam. Note that the original cross section must be used. The distortion of the cross section increases the fully plastic moment for a positive moment.