CLASSICAL MECHANICS (M.Sc. Ist YEAR) 1

Review Of Newtonian Mechanics

We know eq. of motion

$$\vec{F} = m\vec{a}$$ (valid for systems with constant mass)

$$= m\ddot{\vec{r}} = m\frac{d^2\vec{r}}{dt^2}$$

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dr}\left(\frac{d\vec{r}}{dt}\right)$$ (multiplying & dividing by dr)

$$\vec{F} = m\vec{v}\frac{d\vec{v}}{dr}$$

In 1-dimension motion $\vec{v} = \frac{d\vec{x}}{dt}$, $\vec{r} = \vec{x}$

$$\vec{F} = m\vec{v}\frac{d\vec{v}}{dx}$$

$$\boxed{F = mv\frac{dv}{dx}}$$

Now, work done in moving mass 'm' from position $1 \longrightarrow 2$ is

$$W_{12} = \int_1^2 \vec{F}\,d\vec{r}$$

$$= \int_1^2 m\vec{v}\frac{d\vec{v}}{dr}\,d\vec{r}$$

$$= m\int_1^2 \vec{v}\,d\vec{v}$$

$$= m\left|\frac{v^2}{2}\right|_1^2$$

$$W_{12} = \frac{1}{2}m(v_2^2 - v_1^2) \quad \text{— (A)}$$

For conservative force $\vec{F}$

$$\vec{\nabla}\times\vec{F} = 0$$

the above relation is true only if

$$\vec{F} = -\vec{\nabla}\phi$$

where ϕ = potential fn

$$\Rightarrow \quad \vec{\nabla}\times\vec{F} = \vec{\nabla}\times\vec{\nabla}\phi = 0$$

So. $W_{12} = \int_1^2 \vec{F}\, d\vec{r} = -\int_1^2 \vec{\nabla}\phi \cdot d\vec{r}$ $\left\{ \begin{array}{l} \text{Remember} \\ d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} \end{array} \right.$

$$= \int_1^2 \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) \quad (\phi = \text{is f of } x,y,z)$$

$$= \int_1^2 d\phi = (\phi_2 - \phi_1) \text{ --- (B)}$$

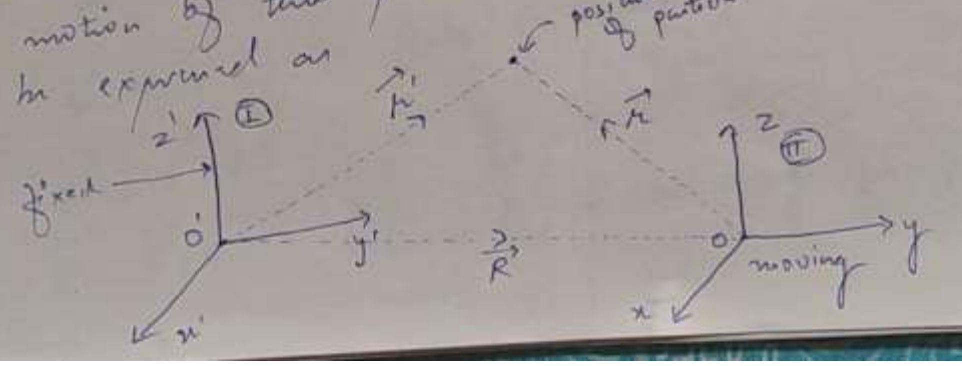
$\Rightarrow$ from eq (A) & (B)

$$\boxed{\frac{1}{2}mv_1^2 + \phi_1 = \frac{1}{2}mv_2^2 + \phi_2}$$

$\Rightarrow$ When a system of constant mass 'm' move in a conservative field 'F' (or force) from position (1) to position (2) then total mechanical energy (K.E + P.E) at position (1) is same as on position (2)

Discussion About Inertial Frame :-

Let us consider a particle moving with force F. The motion of this particle in two inertial frames can be expressed as



Now we can write

$$\vec{r}' = \vec{R} + \vec{r}$$

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt}$$

Now as frame (I) is fixed & frame (II) is moving w.r.t (I)

$$\dot{\vec{r}}' = \vec{V} + \dot{\vec{r}}$$

$$\ddot{\vec{r}}' = \dot{\vec{V}} + \ddot{\vec{r}}$$

if V is constant, then $\dot{\vec{V}} = 0$

$$\ddot{\vec{r}}' = \ddot{\vec{r}} \quad (\ddot{r} = \text{acceleration})$$

so

$\Rightarrow$ If frame (II) is moving with constant velocity w.r.t frame (I), then eq. of motion of particle will remain invariant w.r.t frame of reference.

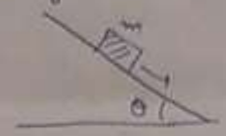
Different types of forces

1) $\vec{F} =$ constant
2) $\vec{F} = \vec{F}(t)$ (like $F = F_0 \cos\omega t$)
3) $\vec{F} = \vec{F}(r)$ ($\vec{F} = \frac{\vec{r}}{r^3}$, central force)
4) $\vec{F} = \vec{F}(\vec{v})$ ($\vec{F} = k\vec{v}$, viscous drag force)

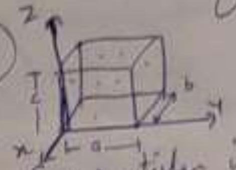
CONSTRAINTS

Anythings restricts the motion of a particle is a constraint

Ex:-

(I) 

mass on an angular plane eq. of motion will be $ax+by+cz=p$

(II) 

Gas particles in a box.

$0 \le x \le -b$
$0 \le y \le a$
$0 \le z \le c$
for particle lying on yz plane

(III) 

pendulum motion

$x^2+y^2=l^2$ for rigid rod

$x^2+y^2 \le l^2$ for string

Constraints which can be expressed in equations are called holonomic constraints.

In above examples case (I) & (III) are holonomic constraints

Case (II) is non-holonomic constraints where constraints are expressed as inequalities & not equations.

Holonomic Constraints

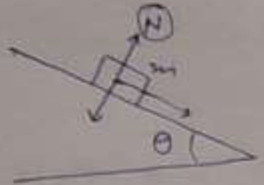
- Rheonomic (depend on time)
- Scleronomic (independent of time) Ex:- Case (III)

$$\vec{F_i} = \dot{\vec{p_i}} \rightarrow \vec{F_i} - \dot{\vec{p_i}} = 0 \rightarrow \sum_i (\vec{F_i} - \dot{\vec{p_i}}) dr_i = 0$$

force — momentum for i^{th} particle

$$\vec{F_i}^{ext} + \vec{F_i}^{int} = \dot{\vec{p_i}}$$

F^{ext} = External force
F^{int} = Internal force



for a man on inclined plane (N) is the normal reaction force which contributes for $\vec{F_i}^{int}$

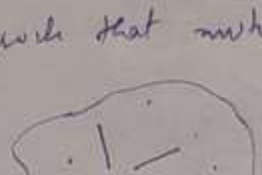
If there is friction present, it will also add to $\vec{F_i}^{int}$.
However, if we consider no friction force, $\vec{F_i}^{int}$ is only due to (N) which is a constraint.

$$\vec{F_i}^{ext} + \vec{F_i}^{int} = \dot{\vec{p_i}} = m_i \frac{d^2 x_i}{dt^2}$$

To solve problem like above, we need to know F information about F_i^{ext}, F_i^{int}, m_i, to determine $x_i(t)$

But there are problems sometime, where we don't know F_i^{int}. Ex:- Rigid body problem

Rigid body is a collection of particles such that mutual distance between them is fixed.



i.e $(\vec{r_i} - \vec{r_j})^2 = C_i$ $\quad C$ = constant

take derivative $\quad r_{ij} = (r_i - r_j)$

$2\vec{r_{ij}} \cdot d\vec{r_{ij}} = 0$

or $F_{ij} \cdot dr_{ij} = 0$ for F to be central force

⇒ Work done in a rigid body is always zero.

Forces of constraint in holonomic constraints systems do not perform any work. (as they are acting ⊥ to motion in most cases)

⇒ For, Holonomic + Scleronomic constraints

$$\vec{r} \perp \vec{F}$$

so $$dW = \vec{F} \cdot \vec{dr} = 0$$

Also For, Holonomic + Rheonomic Constraints

$$\vec{r} \text{ not } \perp \vec{F}$$

so $$\vec{dW} = \vec{F} \cdot \vec{dr} \neq 0$$

But we want work done (dW) to be zero for all holonomic systems.

This can be achieved by <u>VIRTUAL DISPLACEMENT (VD)</u>

and by <u>VIRTUAL WORK (VW)</u>

VD is

1) Displacement consistent with constraint forces
2) There is no passage of time

Consider a pendulum of length $l(t)$ (which is fn of time).

At any instant the pendulum is at position as shown.

Since, at this instant time is not considered to be moving

$l(t) = \text{constant}$

This is holonomic + rheonomic system i.e why $l(t)$

And with $l(t) = \text{constant}$, consider there is small displacement (virtual) δl.

Now, since $l(t) = \text{constant}$, the displ. δl will be along the path of circle of fixed radius.

i.e δl will always be $\perp$ to tension T in the string

In rheonomic constraint $l(t)$ is changing with time, so the motion of bob will not be in a circle of fixed radius but spirally reducing circle.

i.e displ is always $\perp$ to force of constraint.

Virtual Work (VW) = Force × displ (virtual)

$\therefore$

$$\delta W = F^c \cdot \delta l$$

$$\delta W = T \cdot \delta l$$

Since $T \perp \delta l$

$$\boxed{\delta W = 0}$$

$\Rightarrow$ Virtual displ. leads to virtual work which is always zero.

With virtual displ, the work done in Reheonomic constraint systems can be zero.

---

we know $\vec{F} = \dot{\vec{p}}$ $\quad p =$ momentum

for system of particles

$$\sum_i \vec{F_i} = \sum_i \dot{\vec{p_i}}$$

$$\sum_i (\vec{F_i} - \dot{\vec{p_i}}) = 0$$

$\delta r_i =$ virtual displ.

or $\quad \sum_i (\vec{F_i} - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0$

Now $\quad \vec{F_i} = \vec{F}^a + \vec{F}^c$ $\quad \vec{F}^a =$ applied force, $\vec{F}^c =$ force of constraints

$$\sum_i (\vec{F}^a + \vec{F}^c - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0$$

but $\vec{F}^c \cdot \delta \vec{r_i} = 0$ (virtual work)

$$\Rightarrow \boxed{\sum_i (\vec{F}^a - \dot{\vec{p_i}}) \cdot \delta \vec{r_i} = 0} \rightarrow \text{D'Alembert's Principle}$$

$\Rightarrow$ we don't have to worry about force of constraints & work done by them. Only work done by applied forces have significance.

$$\boxed{\sum_{i=1}^{3N} (F_i^a - \dot{p}_i) \cdot \delta x_i = 0} \quad i = 1, 2, 3 \ldots 3N$$

Generalised Coordinates :-

Let us consider a system of N ~~point~~ particles.
So no. of co-ordinates required to specify N particles is 3N.
Let the system have K constraints. So the effective no. of co-ordinates will be 3N-K.

Suppose $3N-K = n$

Let this n no. of independent co-ordinates can be written as

$q_1, q_2, q_3 \cdots q_n$

Now, these new set of co-ordinates $q_1, q_2 \cdots q_n$ should have relation the $(3N-K)$ co-ordinate.

like
$$r_1 = r_1(q_1\, q_2 \cdots q_n)$$
$$r_2 = r_2(q_1\, q_2 \cdots q_n)$$
$$r_3 = r_3(q_1\, q_2 \cdots q_n)$$
$$\cdots$$
$$r_{3N} = r_{3N}(q_1, q_2 \cdots q_n)$$

We know D'Alembert's Principle

$$\sum_{i=1}^{3N} (\vec{F_i}^a - \dot{\vec{p_i}}) \cdot \delta x_i = 0$$

where x_i are standard co-ordinates

We would like to reduce (express) D'Alembert's principle in terms of generalized co-ordinates $3N - K = n$ where $n = q_1 q_2 \cdots q_n$ independent variables. So that

$\boxed{K = \text{constraints}}$

$$C_1 x_1 + C_2 x_2 + C_3 x_3 \cdots C_n x_n = 0$$

if and only if $x_1, x_2, \ldots x_n$ are independent & $C_1, C_2 \ldots C_n = 0$

$\boxed{\text{Here } C_i = (\vec{F_i} - \dot{\vec{p_i}})}$

Now we know for holonomic constraints, we can express them in terms equations. Hence we can express constraint K as $\phi_\alpha (x_1 x_2 \cdots x_{3N}, t) = 0$

$\alpha \to 1 \text{ to } K$

we can write $d\phi_\alpha = 0$

$$\text{or} \quad \sum_{i=1}^{3N} \frac{\partial \phi_\alpha}{\partial x_i} dx_i + \frac{\partial \phi_\alpha}{\partial t} dt = 0$$

for virtual displ. δx_i

$$\sum_{i=1}^{3N} \frac{\partial \phi_\alpha}{\partial x_i} \delta x_i + \frac{\partial \phi_\alpha}{\partial t} \delta t = 0$$

$$\delta \phi_\alpha + 0 = 0$$

where $\delta \phi_\alpha = \frac{\partial \phi_\alpha}{\partial x_i} \delta x_i$ & $\frac{\partial \phi_\alpha}{\partial t} \delta t = 0$

$\Rightarrow$

or $\lambda_\alpha \delta \phi_\alpha = 0$ λ_α = arbitrary constant

Now, since above quantity is a null quantity we can add it to the D'Alembert's principle without loosing ~~any~~ any generality

i.e $\sum_{i=1}^{3N} \left[(F_i^a - \dot{p}_i) + \lambda_\alpha \delta \phi_\alpha \right] \delta x_i = 0$

or $\sum_{i=1}^{3N} \left(F_i^a - \dot{p}_i + \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_i} \right) \delta x_i = 0$

Now above eq. can be expressed as

$C_1 \delta x_1 + C_2 \delta x_2 + C_3 \delta x_3 \dots + C_{3N} \delta x_{3N} = 0$

Since $\delta x_1, \delta x_2 \dots \delta x_{3N}$ are not independent, we can not say $C_1, C_2 \dots C_{3N}$ vanishes independently.

Now if we can adjust λ_α in such a way that K no. of values vanishes automatically, then we have reduced 3N co-ordinates to 3N-K co-ordinates. (which we want to achieve)

where $3N - K = n$ n = independent parameters

Now we have n no. of independent parameters each ~~seperat~~ seperatly vanishes.

i.e $F_i^a - \dot{p}_i + \sum_\alpha \lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_i} = 0$

The relation is called Lagrange's Eq. of I^{st} Kind

& λ_α = Lagrangian Undetermined Multiplier

---

we Know D'Alembert's Principle

$$\sum_{i=1}^{3N} (F_i - \dot{p}_i)\,\delta x_i = 0$$

Now first term in above eq. we can write

$$F_i\,\delta x_i = \sum_i m_i \ddot{x}_i\,\delta x_i$$

where $\delta x_i = \frac{\partial x_i}{\partial q_k}\delta q_k$

$$= \sum_{i,k} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_k}\delta q_k$$

$$= \frac{d}{dt}\left(m_i \dot{x}_i \frac{\partial x_i}{\partial q_k}\delta q_k\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) \quad (A)$$

Now we can write

$$dx_i = \frac{\partial x_i}{\partial q_k} dq_k + \frac{\partial x_i}{\partial t} dt$$

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial x_i}{\partial t}$$

$$\dot{x}_i = \frac{\partial x_i}{\partial q_k}\dot{q}_k$$

$$\boxed{\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}}$$

put in eq. (4)

$$= \frac{d}{dt}\left(m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial q_k}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)$$

T= Kintetic Energy

CENTRAL FORCES

Family of force field with following properties

→ Force field pointing towards or out of a fixed point

→ Magnitude of force along the line towards or away from fixed point

$|\vec{F}| = \vec{F}(\vec{r})$ i.e magnitude of force will be fn of dis r

→ Such forces are conservative by nature. (Energy, E = const)

→ Angular momentum is conserved i.e $\vec{L}$ = const

→ Motion is planar i.e motion of particle under the influence of central force only, then is planar or confined to a plane.

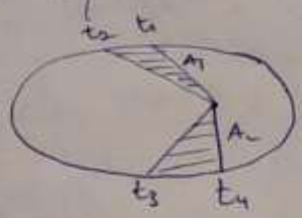
Examples :- 1) Gravitational force
2) Coulomb attraction / repulsion OR Electrostatic force
3) Spring force OR force due to a spring

KEPLER'S LAWS

1) Path / orbit of each planet is an ellipse with sun at one foci

2) The line joining the sun with planet, sweeps out equal area in equal interval of time

$$\frac{A_1}{t_2 - t_1} = \frac{A_2}{t_4 - t_3}$$



3) If a planet has time period T, then T is related to semi-major axis of ellipse by relation

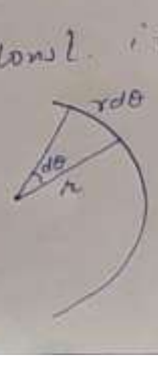
$$T^2 \propto a^3$$

Proof of Kepler's Laws

1) In case of inverse sq. force law i.e $F \propto \frac{1}{r^2}$ the most general case of closed orbit is ellipse.

i.e first law is correct

2) IInd law states that $\frac{dA}{dt} = $ const. i.e areal velocity is constant



$$dA = \frac{1}{2} r \cdot r d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

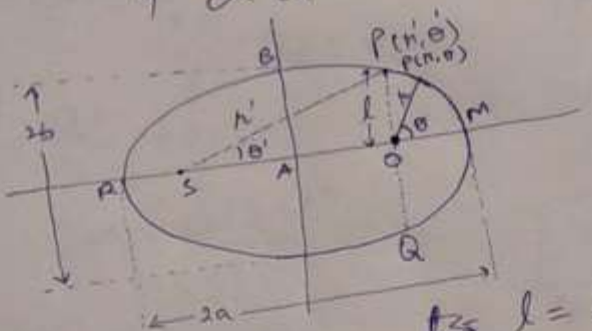
Now $r^2 \dot{\theta}$ for any central orbit is a constant equal to $\frac{L}{m}$

$$\frac{dA}{dt} = \frac{1}{2} \frac{L}{m} = \text{Constant}$$

$$\frac{dA}{dt} = \text{constant}$$

3) Third law states that

$$T^2 \propto a^3$$ where a = semi-major axis



l = semi-latus rectum

$PQ = 2l$ = latus rectum

$$\frac{L}{r} = 1 + \epsilon \cos\theta$$

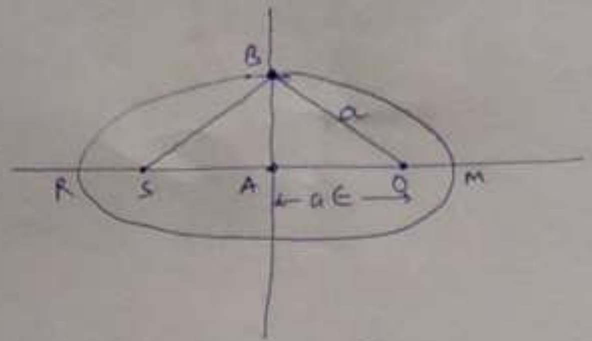
for $\epsilon = 0 \rightarrow$ circle

$0 < \epsilon < 1 \rightarrow$ ellipse

$\epsilon = 1 \rightarrow$ parabola

$\epsilon > 1 \rightarrow$ hyperbola

At M, $r + r' = 2a$



$OB + SB = 2OB = 2a$

$OB = a$

$OA = AM - OM$

$OA = a - OM \longrightarrow$

$OM = \frac{l}{1+\epsilon}$

|||ly $OR = \frac{l}{1-\epsilon}$

$OM + OR = 2a$

$\frac{l}{1+\epsilon} + \frac{l}{1-\epsilon} = 2a$

$2a = \frac{2l}{1-\epsilon^2}$

$$\boxed{a = \frac{l}{1-\epsilon^2}}$$

Now $OM = \frac{l}{1+\epsilon}$

and $a = \frac{l}{1-\epsilon^2}$

$OM = \frac{a(1-\epsilon^2)}{1+\epsilon}$

$OM = a(1-\epsilon)$

$\Rightarrow$ $OA = a - a(1-\epsilon)$

$$\boxed{OA = a\epsilon}$$

so, $AB = \sqrt{a^2 - a^2\epsilon^2}$

$AB = a\sqrt{1-\epsilon^2}$

$b = a\sqrt{1-\epsilon^2}$

Now $T = \frac{\text{area}}{\text{areal velocity}}$

$T = \frac{\pi a b}{L/2m}$

$T = \frac{\pi a^2 \sqrt{1-\epsilon^2}}{L/2m}$

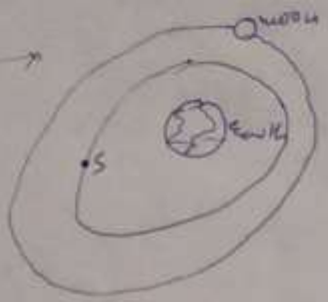
$T^2 = \frac{4m^2\pi^2}{L^2} a^4 (1-\epsilon^2)$

Now $l = a(1-\epsilon^2)$

$$(1-\epsilon^2) = \frac{l}{a}$$

$$T^2 = \left[\frac{4m^2\pi^2}{L^2} l\right] a^3$$

$$\boxed{T^2 \propto a^3}$$



$T_m \sim 28$ days

s = satellite

$$T^2 \propto a^3$$

$$\left(\frac{T_M}{T_s}\right)^2 = \left(\frac{a_m}{a_s}\right)^3$$

If we know three parameters, fourth one can be calculated