Scalar ↳

A Physical Quantity which has only magnitude is called as a scalar. ex. Speed, Mass.

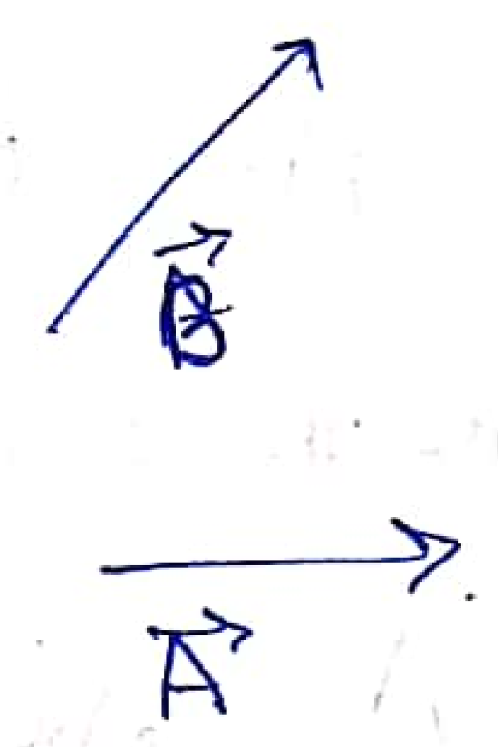
Vector:-

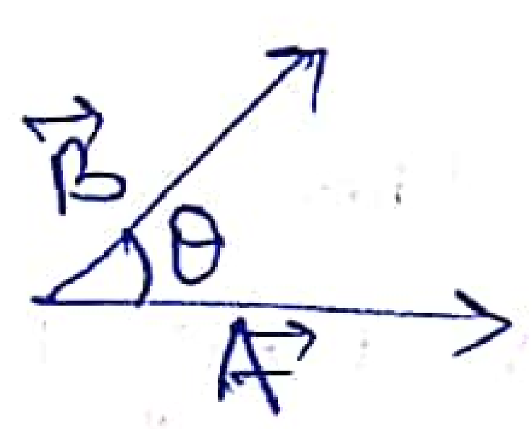
A Physical Quantity which has both magnitude and direction is called Vector. Example Velocity, Acceleration

Velocity

Dot Product of two Vector:-

$$\vec{A}\cdot\vec{B} = AB\cos\theta \quad \text{--- (1)}$$





Cross Product of two Vector:-

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n} \quad \text{--- (2)}$$

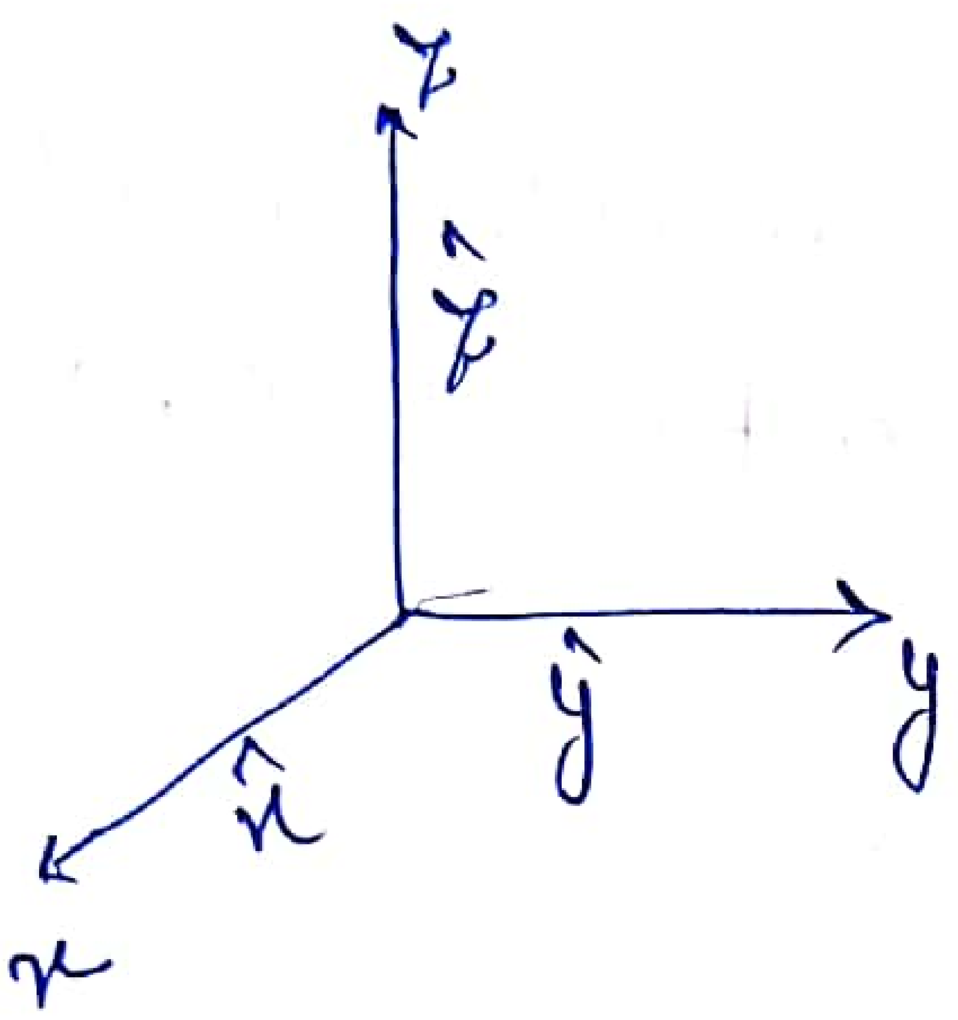
where $\hat{n}$ is a unit vector pointing prependicular to the plane $\vec{A}$ & $\vec{B}$.

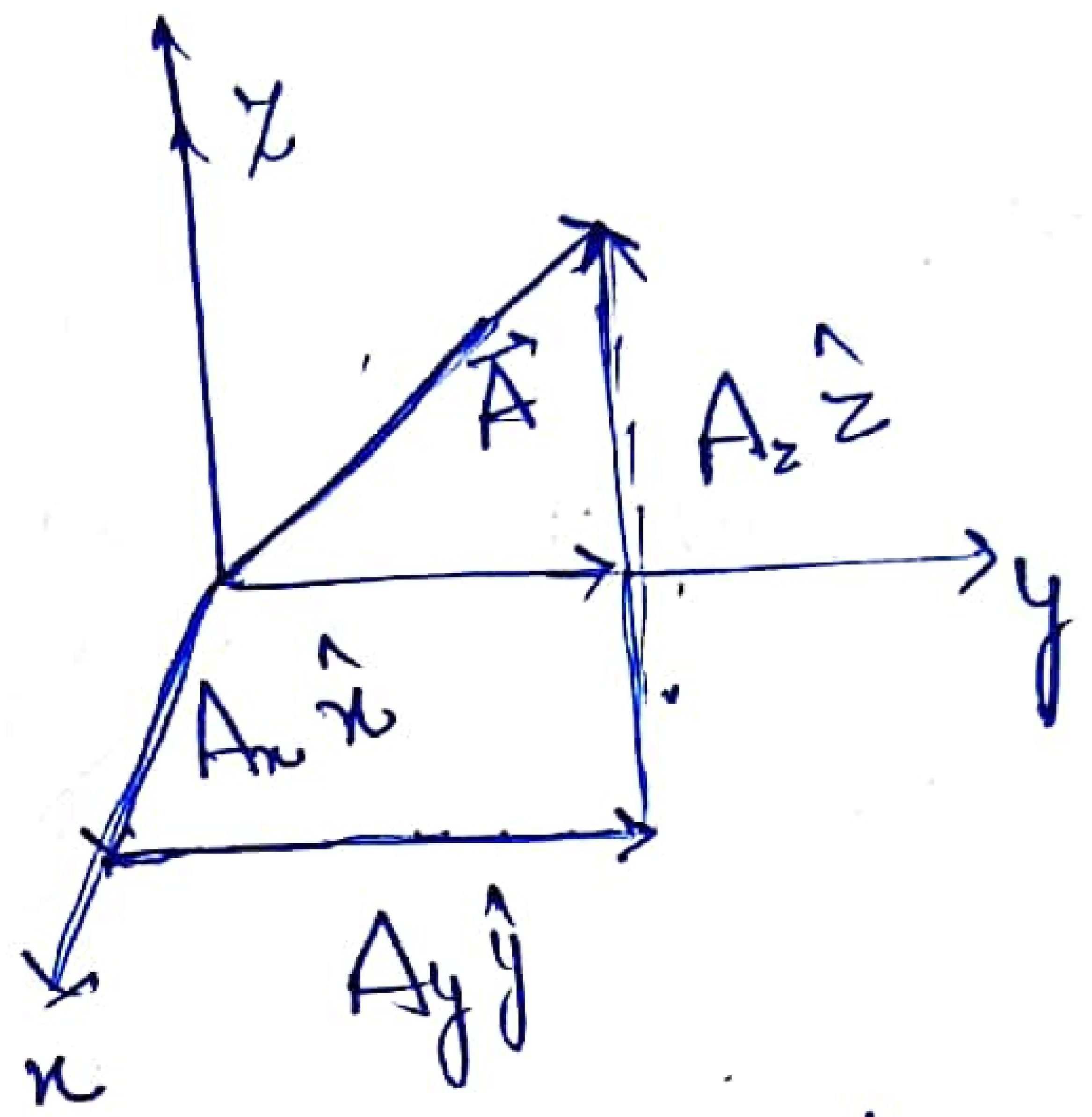
$\vec{A}\times\vec{B}$ in determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{--- (3)}$$

Vector components form :-

In ~~para~~ practice, if we have choosen Cartesian Co-ordinates x, y, z. Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be unit vectors parallel to the x, y and z axes, respectively. An arbitrary vector $\vec{A}$ can be expanded in terms of these basis vectors.





$\vec{A}$ in component form

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

A_x, A_y, A_z are the components of $\vec{A}$.

$$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2$$

$$A^2 = A_x^2 + A_y^2 + A_z^2 \Rightarrow \boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}} \quad (4)$$

Note:

(i) $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$

(ii) $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$

(iii) $\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0$

(IV) $\hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$,

$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = +\hat{x}$

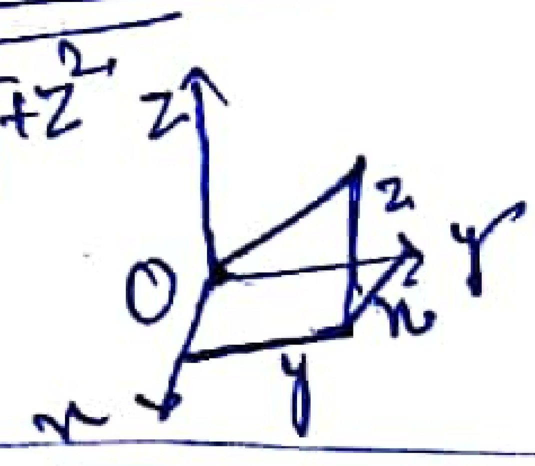
$\hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}$

The vector to that point from the origin (O) is called the position vector

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\hat{r} = \frac{\vec{r}}{r}$$



$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}} \quad (5)$$

$\hat{r} \rightarrow$ unit vector pointing radially outward.

The infinitesimal displacement vector from (x, y, z) to $(x+dx, y+dy, z+dy)$ is

$$d\vec{l} = d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} \quad (6)$$

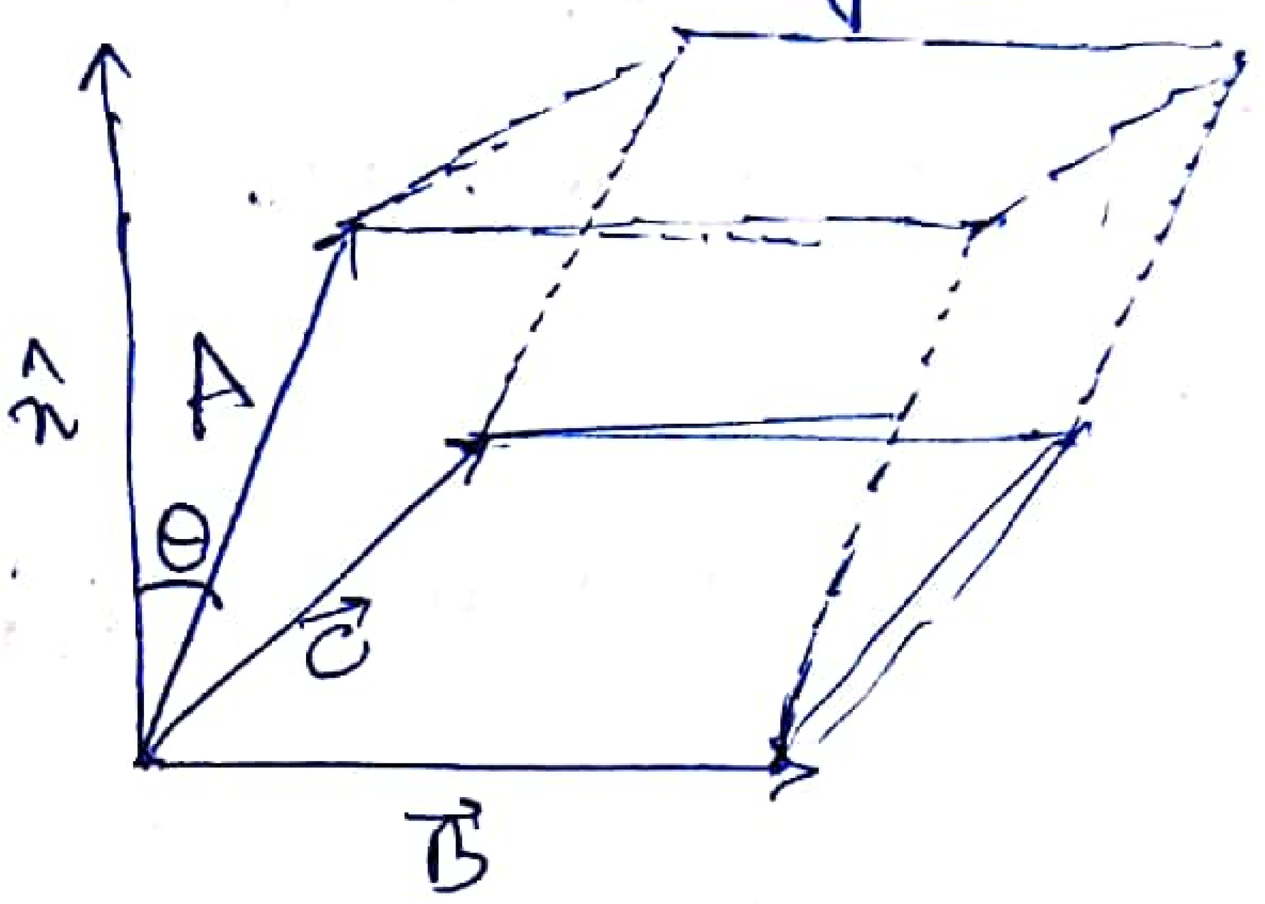
Triple Product

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a triple product

(i) Scalar triple product :-

$A\cdot(B\times C)$. Geometrically, $|A\cdot(B\times C)|$ is the volume of the parallelopiped generated by A, B & C, since $|B\times C|$ is the area of the base, and $|A\cos\theta|$ is the Altitude shown in figure.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$



$\vec{A}\cdot(\vec{B}\times\vec{C})$ in component form

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad \text{--- (7)}$$

- Notes: Dot or Cross can be interchanged

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

But $(\vec{A}\cdot\vec{B})\times\vec{C}$ is a meaningless expression.

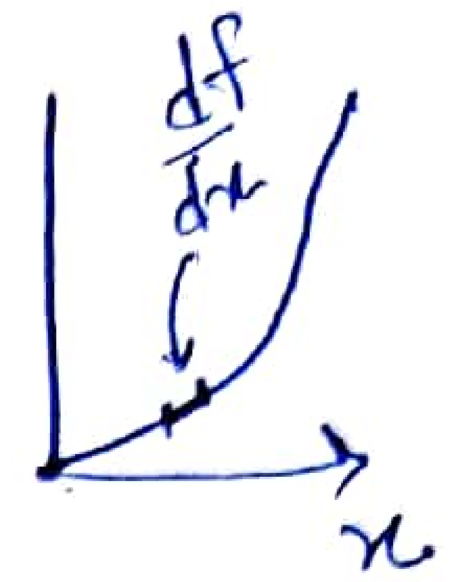
(II) Vector Triple product :-

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Simplified form $\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = BAC - CAB}$ — (8)

Ordinary derivatives :-

Suppose we have a function of one variable: $f(x)$



$$df = \left(\frac{df}{dx}\right) dx$$

df/dx tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx.

In other word, If we increment x, by an infinitesimal amount dx, then f changes by an amount df

The derivative df/dx is the slope of the graph of f versus x.

Vector Differential Operator {∇ 'Del'}: It is denoted by '∇' and defined as

$$\boxed{\nabla = \text{Del} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}} \quad \text{— (9)}$$

operator ∇ can act:

(i) On a scalar function T: ∇T (The gradient)

(ii) On a vector function $\vec{A}$: $\nabla \cdot \vec{A}$ (The divergence)

(iii) on a vector function $\vec{A}$ via cross product: $\nabla \times \vec{A}$ (The curl)

Gradient :-

Let 'f' is a scalar point function, then the gradient of f is denoted by ∇f or grad f and is defined as $\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$$\nabla \cdot f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \quad \text{— (10)}$$

* The operator gradient is always applied on scalar field and the resultant will be a vector. i.e. the operator gradient converts a scalar field into a vector field.

Suppose, now, that we have a function of three variables, say the Temperature $T(x, y, z)$ {Remember, Temperature is scalar quantity} in a room. 'T' which depend not on one but on three variables.

From partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz \quad \text{— (11)}$$

This tells us how T changes when we alter all three variables by the ~~infinitesimal~~ an infinitesimal amount dx, dy, dz.

above equation (11) can be written ~~as~~ of a dot product.

$$dT = \underline{\left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= (\nabla T)\cdot(d\vec{l}) \quad \text{— (12)}$$

~~Were~~ where

$$\nabla T \equiv \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

<u>In generalized form</u>

$$\nabla T \equiv \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)$$

$\nabla T \rightarrow$ is the gradient of T.

$\nabla T \rightarrow$ is a Vector quantity

<u>Geometrical Interpretation of the Gradient:-</u>

Like any vector, the gradient has magnitude and direction.

Let's rewrite the dot product of eqn (12)

$$dT = \nabla T\cdot d\vec{l} = |\nabla T|\,|d\vec{l}|\cos\theta$$

$\theta \rightarrow$ angle between ∇T & $d\vec{l}$. If we fix magnitude $|d\vec{l}|$ and search around in various directions (that is, vary θ), the maximum change in T evidentally occurs when $\theta = 0$.

The magnitude $|\nabla T|$ gives the slope (rate of increase) ~~and~~ along this maximal direction.

Divergence :-

Let $\vec{v} = (v_x \hat{i} + v_y \hat{j} + v_z \hat{k})$ is a Vector point function, then the divergent of $\vec{v}$ is denoted by $\nabla \cdot \vec{v}$ or div $\vec{v}$

$$\nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(v_x \hat{i} + v_y \hat{j} + v_z \hat{k})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

* The operator divergent is always applied on a vector field and resultant will be a scalar.

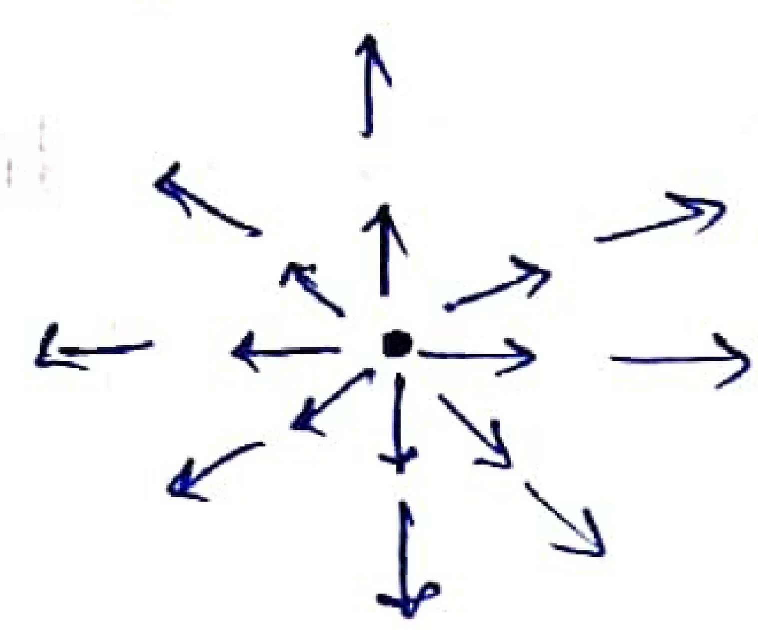
The operator divergent will converts a vector into a scalar.

Solenoidal Vector :-> $\boxed{\text{div } \vec{v} = 0}$ OR $\boxed{\nabla \cdot \vec{v} = 0}$

Geomatrical Interpretation :-> $\nabla \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges)

Positive divergence -> Source or faucet

Negative divergence -> Sink or Drain



Curl of a Vector

Let $\vec{v} = (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})$ is a vector valued function, then curl of vector $\vec{v}$ is denoted by curl $\vec{v}$ and defined as $\nabla \times \vec{v}$

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Note! The operator curl is applied on a vector field

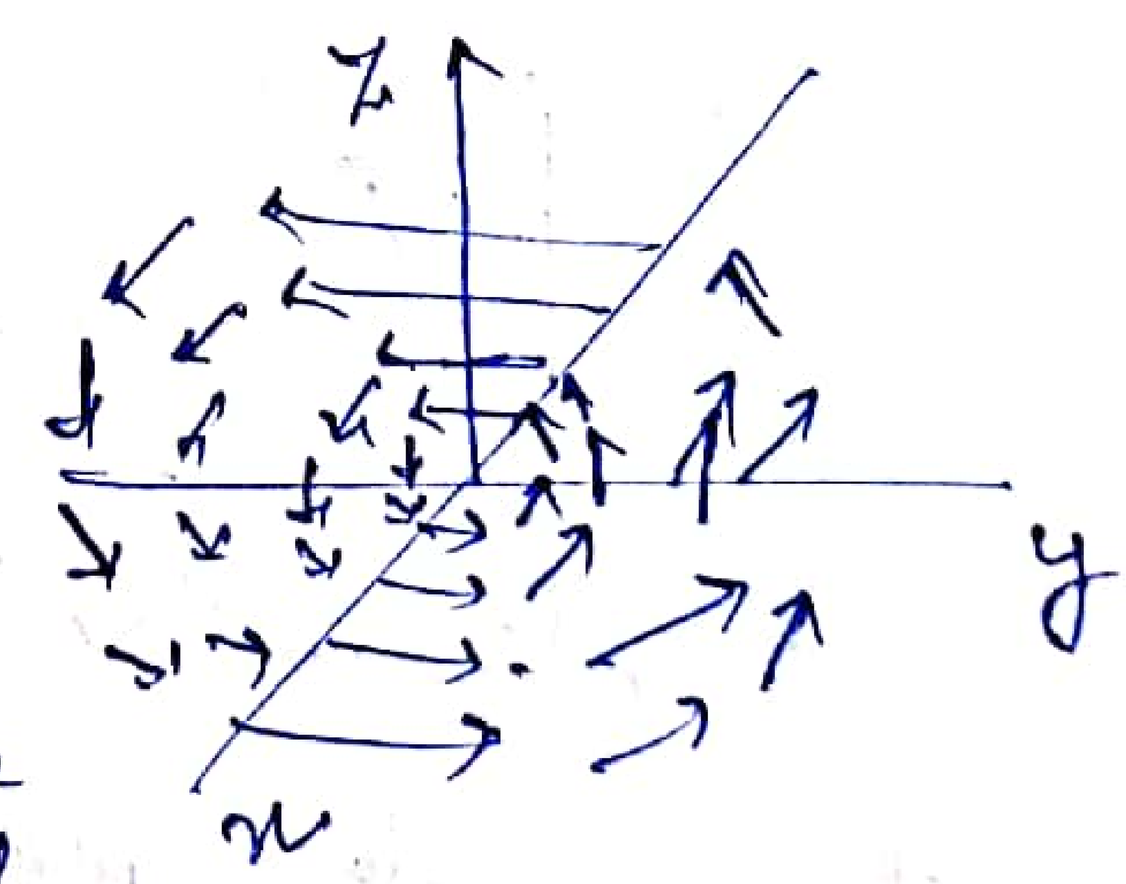
Irrotational field

If $\boxed{\text{curl } \vec{v} = \nabla \times \vec{v} = 0}$

Then $\vec{v}$ is irrotational field vector.

Geometrical Representation!

$\nabla \times \vec{v}$ OR Curl $\vec{v}$ is a measure of how much the vector $\vec{v}$ swirls around the point.
↳ whirl, vortex



function in the figure have a substantial ~~Figure~~ Curl, pointing in the direction of z, as the natural right-hand rule would suggest.

example! Imagine you are standing at the edge of a pond. ~~Float~~ a Float a small paddle wheel, it. If it is starts to rotate, then you placed it at a point of nonzero curl. A whirlpool would be a region of large curl.

Second Derivatives :-

As we can understood the gradient, the divergence and the curl are only first derivatives. We can make with ∇ (del) : by applying ∇ twice.

We can construct five species of second derivatives

The ~~diver~~ gradient ∇T is a vector, so we can take the divergence & curl of it

(i) Divergence of gradient : $\nabla\cdot(\nabla T)$.

(ii) Curl of gradient : $\nabla\times(\nabla T)$.

Remember $T \rightarrow$ is a scalar function

(i) $\nabla\cdot(\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$

$$\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

This ∇^2 known (∇^2) Laplacian operator

If we write Laplacian for vector field

$$\nabla^2 v = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$$

(ii) The curl of a gradient is always zero

$$\nabla\times(\nabla T) = 0$$

On solving this expression $[\nabla\times(\nabla T)]$ by determinant method,

$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$, we get zero

(iii) Gradient of divergence : $\nabla \cdot (\nabla v)$

(iv) Divergence of curl : $\nabla \cdot (\nabla \times v) = 0$

(V) Curl of curl : $\nabla \times (\nabla \times v) = \nabla(\nabla \cdot v) - \nabla^2 v$

$$A \times (B \times C) = BAC - CAB$$

There are only two kinds of second derivatives:

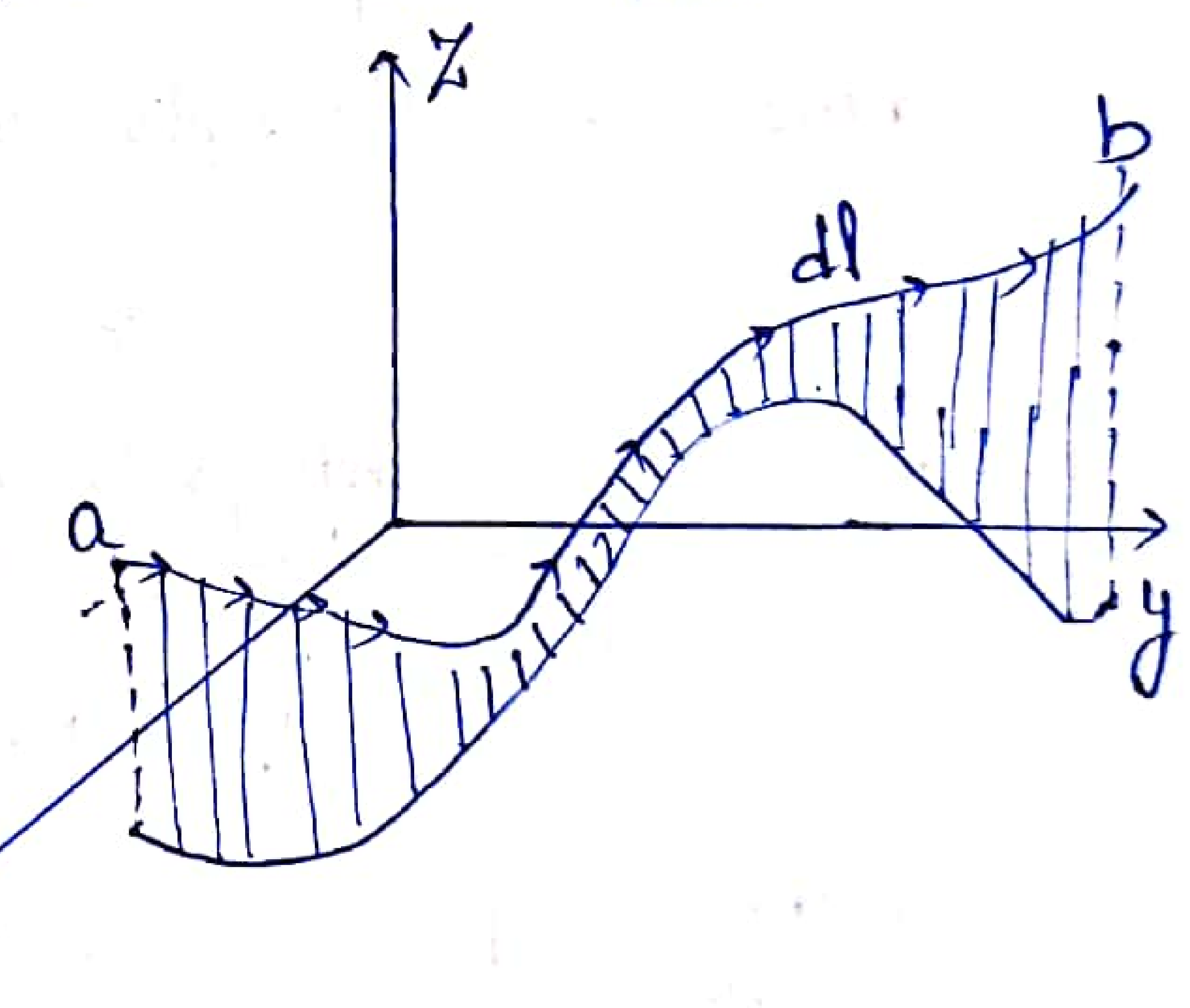
1. The Laplacian (which is fundamental importance)
2. Gradient of divergence (which we rarely encounter)

Integral Calculus :→

(a). Line Integral (or path) :→

A line integral is an expression of the form

$$\int_a^b v \cdot dl$$



Where $\vec{v}$ is a vector function, dl is the infinitesimal displacement vector and the integral is to be carried out along a ~~pat~~ prescribed path P from point a to point b.

If the path in question forms a closed loop (if b = a), put a circle on the integral sign:

$$\oint v \cdot dl.$$

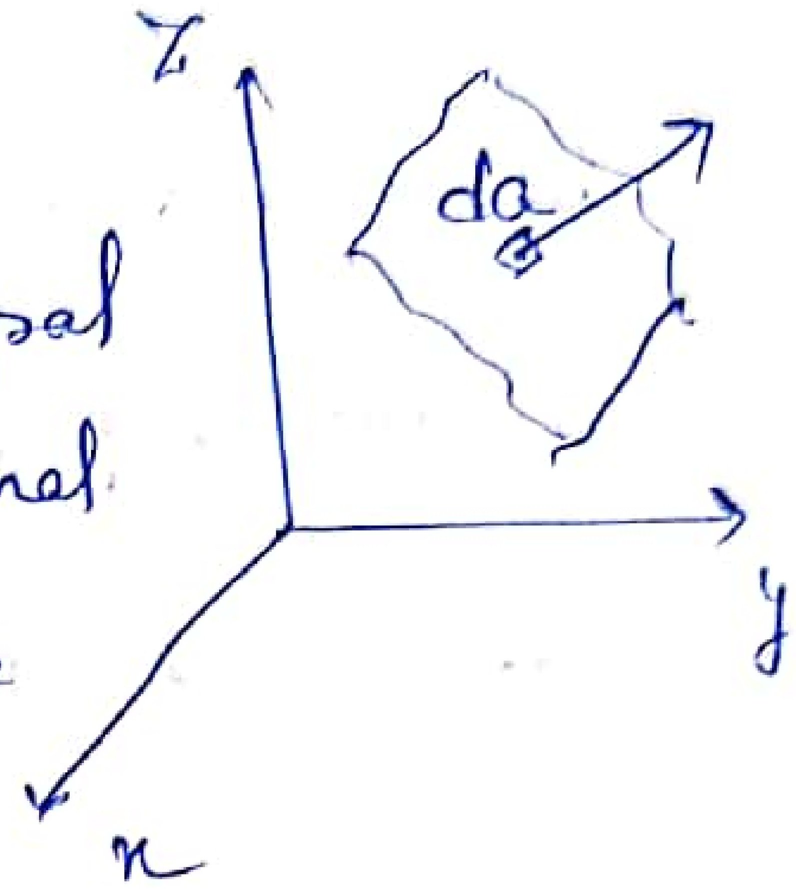
Physical significance :→

The most familiar example of a line integral is the work done by a force F : $W = \int F \cdot dl$.

(b) 1. Surface Integral :- (or flux)

A surface integral is an expression

$$\int_S V \cdot da = \iint V \cdot da$$



Where V is some vector function, and the integral is over a specified surface S. da is an infinitesimal patch of area, with direction perpendicular to the surface.

If the surface is closed ~~loop~~

$$\oint V \cdot da$$

Physical Significance :- If V described the flow of a fluid (mass per unit area per unit time), then $\int V \cdot da$ represents the total mass per unit time passes through the surface - hence the alternative name "flux".

(c) Volume Integrals :-> A volume integral is an expression

$$\int_V T \, d\tau = \iiint T \, d\tau$$

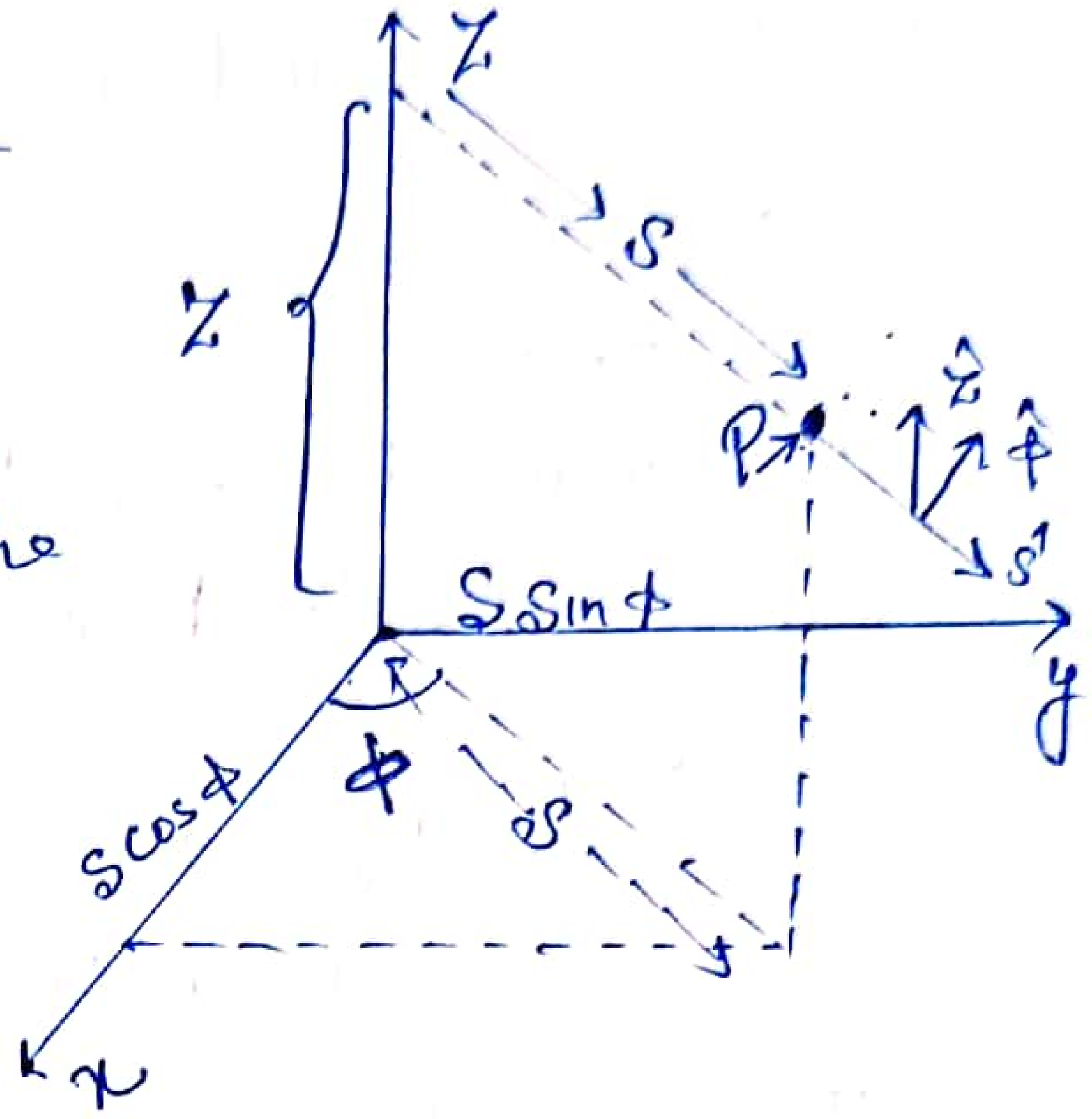
Remember T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian Co-ordinate

Physical Significance

$$d\tau = dx \, dy \, dz$$

For example :- if T is density of a substance, then the volume integral would give the total mass.

Cylindrical Coordinates



The cylindrical coordinates (s, ϕ, z) of a point P as shown in figure.

$s \rightarrow$ is the distance to P from the z-axis

$z \rightarrow$ is the same as cartesian

$\phi \rightarrow$ azimuthal angle
the angle around from x axis.

The relation to cartesian co-ordinates

$$x = s\cos\phi \qquad s^2 = x^2 + y^2$$

$$y = s\sin\phi \qquad \boxed{s = \sqrt{x^2+y^2}}$$

$$z = z \qquad \boxed{\phi = \tan^{-1}(y/x)}$$

unit vector

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \qquad \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\hat{z} = \hat{z}$$

The infinitesimal displacements are

$$dl_s = ds, \quad dl_\phi = s\,d\phi, \quad dl_z = dz$$

$$\boxed{d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}}$$

The volume element is

$$\boxed{d\tau = s\,ds\,d\phi\,dz}$$

s is $0 \rightarrow \infty$, ϕ goes from $0 \rightarrow 2\pi$ & z from $-\infty$ to ∞

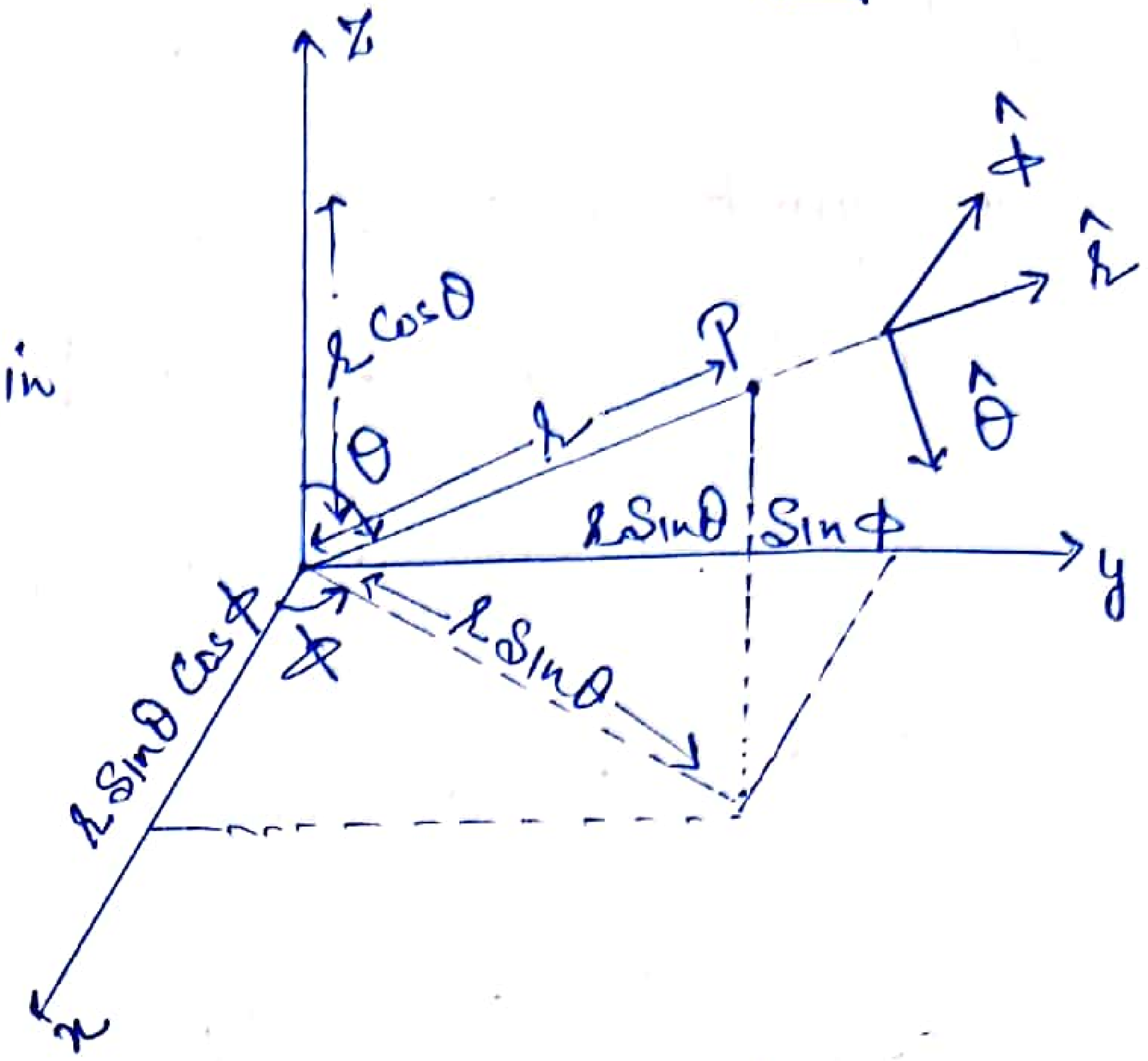
Spherical Coordinates :→

Spherical Coordinates (r, θ, ϕ)

$r \rightarrow$ is the distance from the origin ~~magn~~

$\theta \rightarrow$ Angle down from the z-axis is called polar angle

$\phi \rightarrow$ The angle around from the x-axis is azimuthal Angle



Their relation to cartesian Co-ordinates can be read as

Ranges

$r \rightarrow 0$ to ∞

$\phi \rightarrow 0$ to 2π

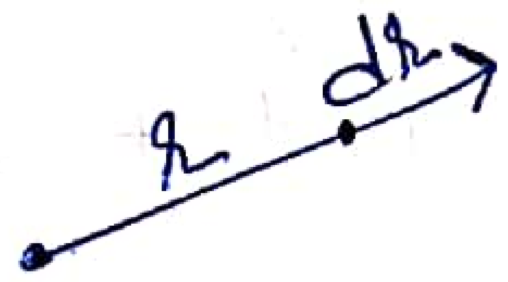
$\theta \rightarrow 0$ to π

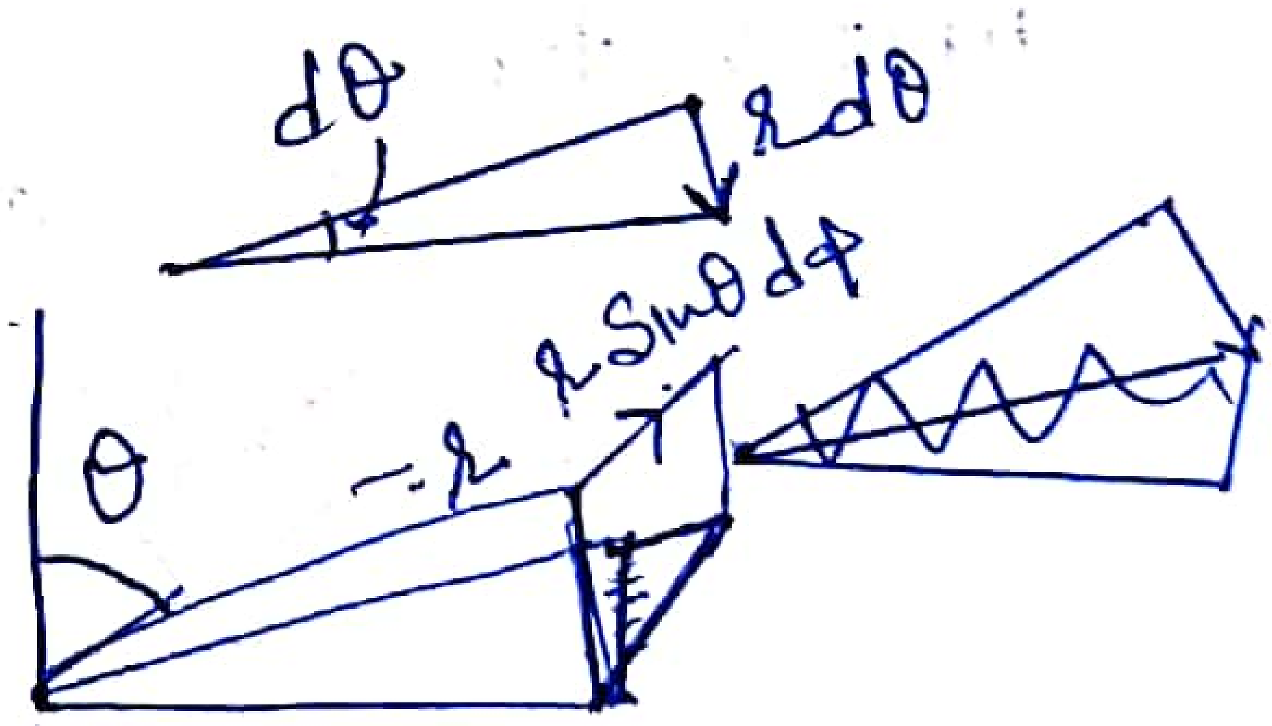
$$x = r\sin\theta \cos\phi$$
$$y = r\sin\theta \sin\phi$$
$$z = r\cos\theta$$

$$r = \sqrt{x^2+y^2+z^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$\theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}$$

An infintesimal displacement in the $\hat{r}$ is dr.

$$dl_r = dr$$
$$dl_\theta = r\,d\theta$$
$$dl_\phi = r\sin\theta\, d\phi$$





Thus the general infinetesimal displacement dl is

$$dl = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

The infinetesimal Volume element $d\tau$, in spherical coordinates

$$d\tau = dl_r\, dl_\theta\, dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

For surface element

$$da_1 = dl_\theta\, dl_\phi\, \hat{r} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$ [r is constant θ, φ vary

$$da_2 = dl_r\, dl_\phi\, \hat{\theta} = r\, dr\, \sin\theta\, d\phi\, \hat{\theta}$$ [xy plane, θ → Constant r, φ vary

Find the Volume of a sphere of Radius R

$$V = \int d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^R r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \frac{R^3}{3} \cdot (2) \cdot (2\pi) = \frac{4\pi}{3} R^3$$

Find the Volume of a cylinder of radius R & height H

$$V = \iiint d\tau = \int_{s=0}^{R} \int_{\phi=0}^{2\pi} \int_{z=0}^{H} s \, ds \, d\phi \, dz$$

$$= \left[\frac{s^2}{2}\right]_0^R \left[\phi\right]_0^{2\pi} \left[z\right]_0^H$$

$$= \frac{R^2}{2} \, 2\pi \, H$$

$$\boxed{V = \pi R^2 H}$$

The Fundamental theorem of Calculus :-

Suppose $f(x)$ is a function of one variable. The fundamental theorem of calculus says:

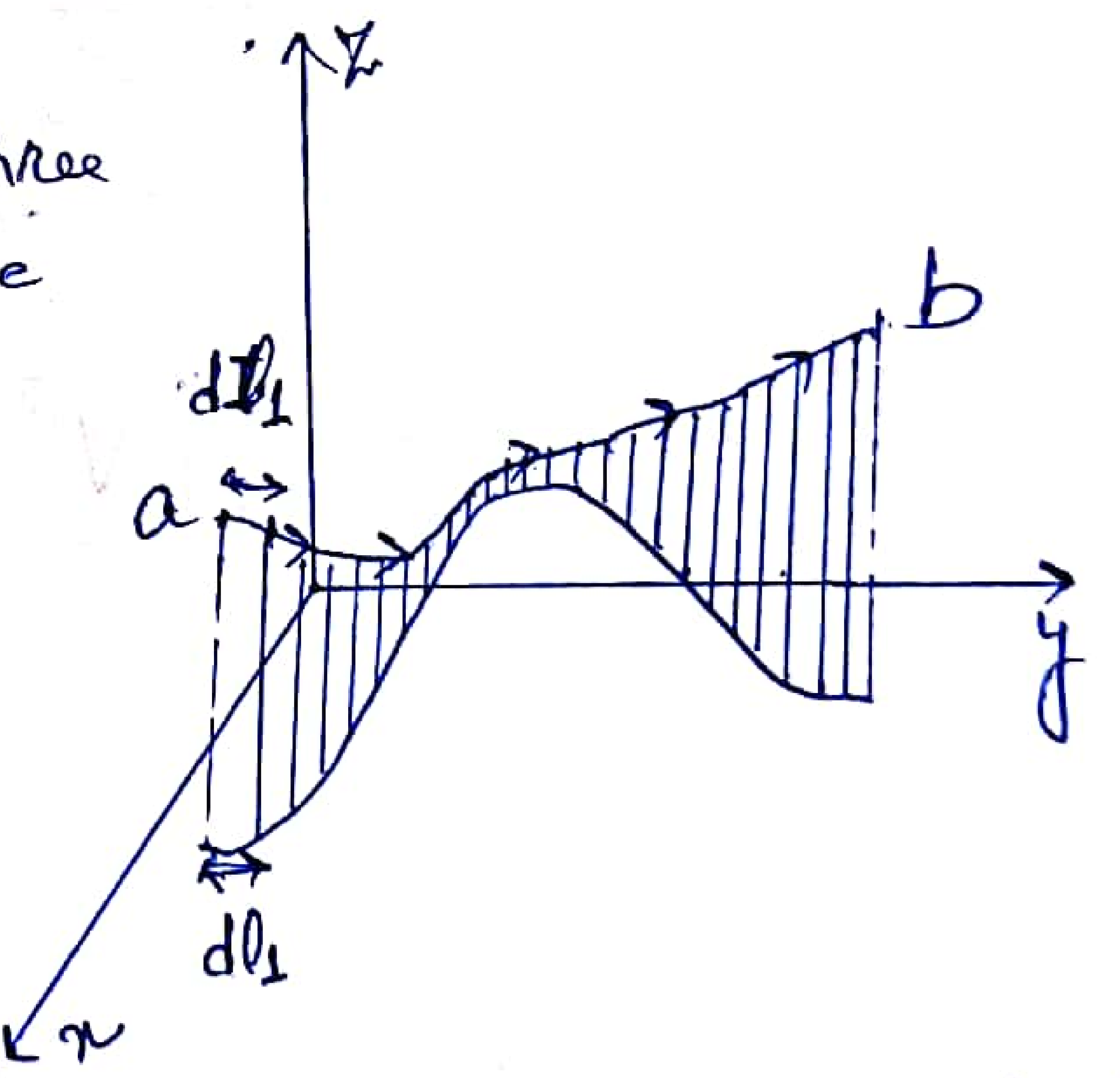
$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a) \qquad (14.1)$$

In another way

$$\int_a^b F(x)\,dx = f(b) - f(a) \qquad \text{where } F(x) = \frac{df}{d(x)}$$

The fundamental theorem tells you how to integrate $F(x)$: a function $f(x)$ whose derivative is equal to F.

Fundamental theorem for gradient :-



Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point a, we move a small distance dl_1.

The function T will change by an amount

$$dT = \nabla T \cdot dl_1$$

Same as now we move a little further by an additional small displacement dl_2. The incremental change in T will be $(\nabla T) \cdot dl_2$. In this manner, ~~we proceed~~ proceeding by infinitesimal steps, we reach to point b.

The Total changes in T in going from a to b (along the path) is

$$\boxed{\int_a^b (\nabla T) \cdot dl = T(b) - T(a)} \qquad (14.2)$$

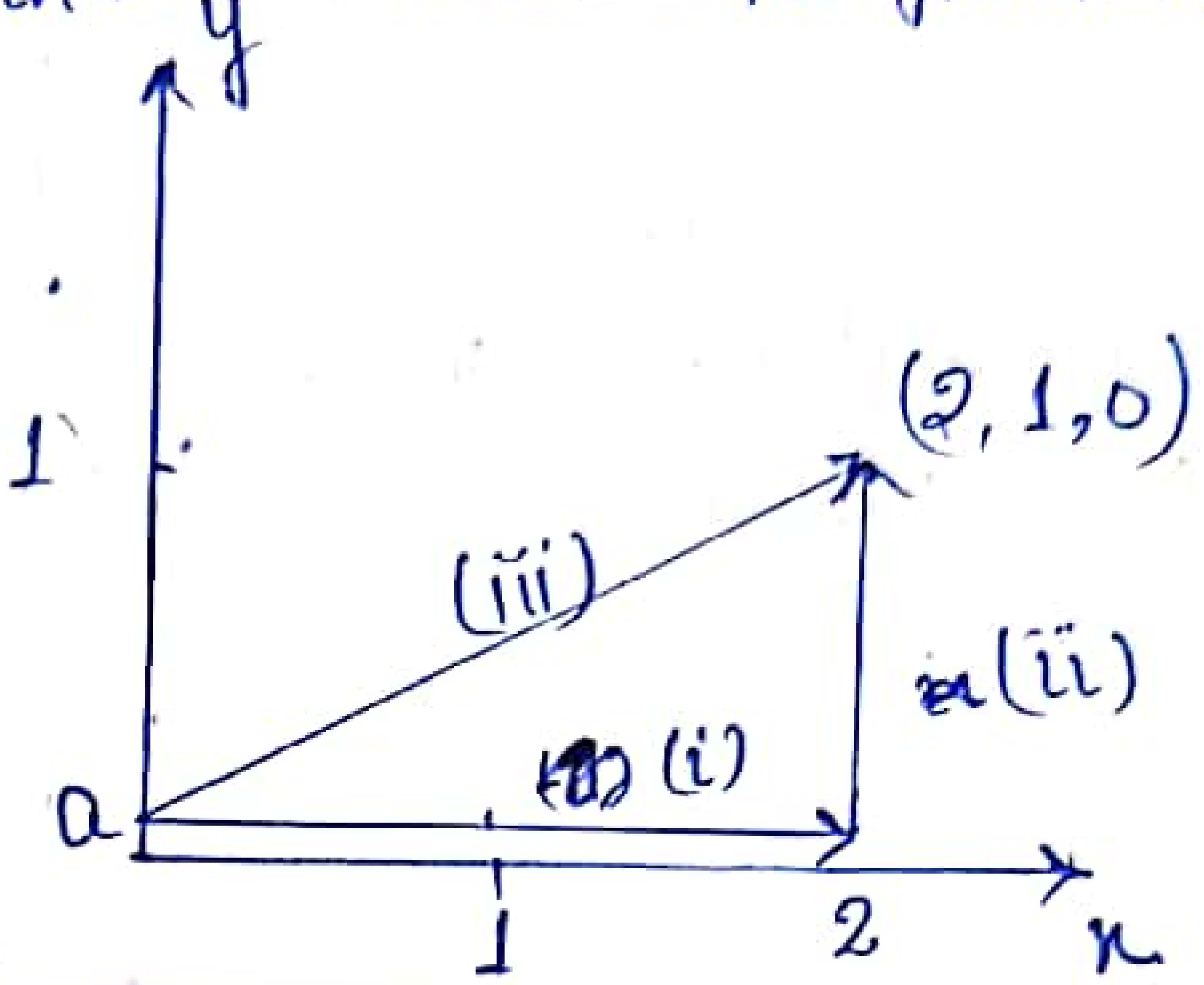
Integral of a derivative is given by the value of function at the boundaries (a & b).

example

Let $T = xy^2$, and take a point 'a' to be the origin $(0,0,0)$ and 'b' the point $(2,1,0)$. Check the fundamental theorem for gradients.

Solution

$$\int_a^b \nabla T \cdot dl \rightarrow \text{Fundm Thm Grad.}$$



$$T = xy^2$$

$$\nabla T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot xy^2$$

$$= y^2\hat{x} + 2xy\,\hat{y} + 0$$

$$\boxed{\nabla T = y^2\hat{x} + 2xy\,\hat{y}}$$

$$dl = dx\,\hat{x} + dy\,\hat{y} + \underbrace{dz\,\hat{z}}_{0}$$

$$\boxed{dl = dx\,\hat{x} + dy\,\hat{y}}$$

Lets go along x-axis (step i) & then step (ii).

Step (i)

$y = 0 \quad dy = 0$

$\nabla T = 0$

$dl = dx\,\hat{x}$

$$\int_{\text{step i}} \nabla T \cdot dl = 0$$

Step ii

$x = 2 \quad dx = 0$

$dl = 0 + dy\,\hat{y} = dy\,\hat{y}$

$\nabla T = y^2\hat{x} + 4y\,\hat{y}$

$$\int_{\text{step ii}} \nabla T \cdot dl = \int_0^1 (y^2\hat{x} + 4y\,\hat{y})\cdot dy\,\hat{y}$$

$$= \int_0^1 y^2 dy\,\underbrace{\hat{x}\cdot\hat{y}}_{0} + \int_0^1 4y\,dy\,\underbrace{\hat{y}\cdot\hat{y}}_{1}$$

$$= \int_0^1 4y\,dy = 4\left[\frac{y^2}{2}\right]_0^1$$

$$= 2$$

The total Integral is $\underline{\underline{2}}$.
This is consistent with fundamental theorem of Gradient.

$$T(b) - T(a) = 2 - 0$$
$$= \underline{\underline{2}}$$

Hence Proved //

Calcula

Step iii Calculated the integral along path iii (straight line from a to b)

$y = mx + c \Rightarrow y = \frac{1}{2}x + 0$

$y = \frac{x}{2} \Rightarrow dy = \frac{1}{2}dx$

$$\nabla T \cdot dl = (y^2\hat{x} + 2xy\,\hat{y})\cdot(dx\,\hat{x} + \tfrac{1}{2}dx\,\hat{y})$$

$$= y^2 dx + xy\,dx = \frac{x^2}{4}dx + \frac{x^2}{2}dx$$

$$= \frac{3x^2}{4}dx$$

$$\int_0^2 \nabla T\cdot dl = \int_0^2 \frac{3x^2}{4}dx = \frac{3}{4\times 3}[x^3]_0^2 = 2$$ //

The fundamental theorem for Divergence :→ OR Gauss Divergence

The fundamental theorem for divergence states that :

Where $\hat{n}$ is outward drawn unit normal vector over surface

$$\int_V (\nabla \cdot \vec{v}) \cdot d\tau = \oint_S \vec{v} \cdot da \cdot \hat{n}$$

$$\boxed{\iiint (\nabla \cdot \vec{v}) d\tau = \iint \vec{v} \cdot da \, \hat{n}}$$

$$\boxed{\iiint (\text{div } \vec{v}) d\tau = \iint \vec{v} \cdot da \, \hat{n}}$$ Known as Gauss divergence ~~theor.~~ theorem

It says that the integral of a derivative over a region (Volume V) is equal to the value of function at the boundary (Surface S)

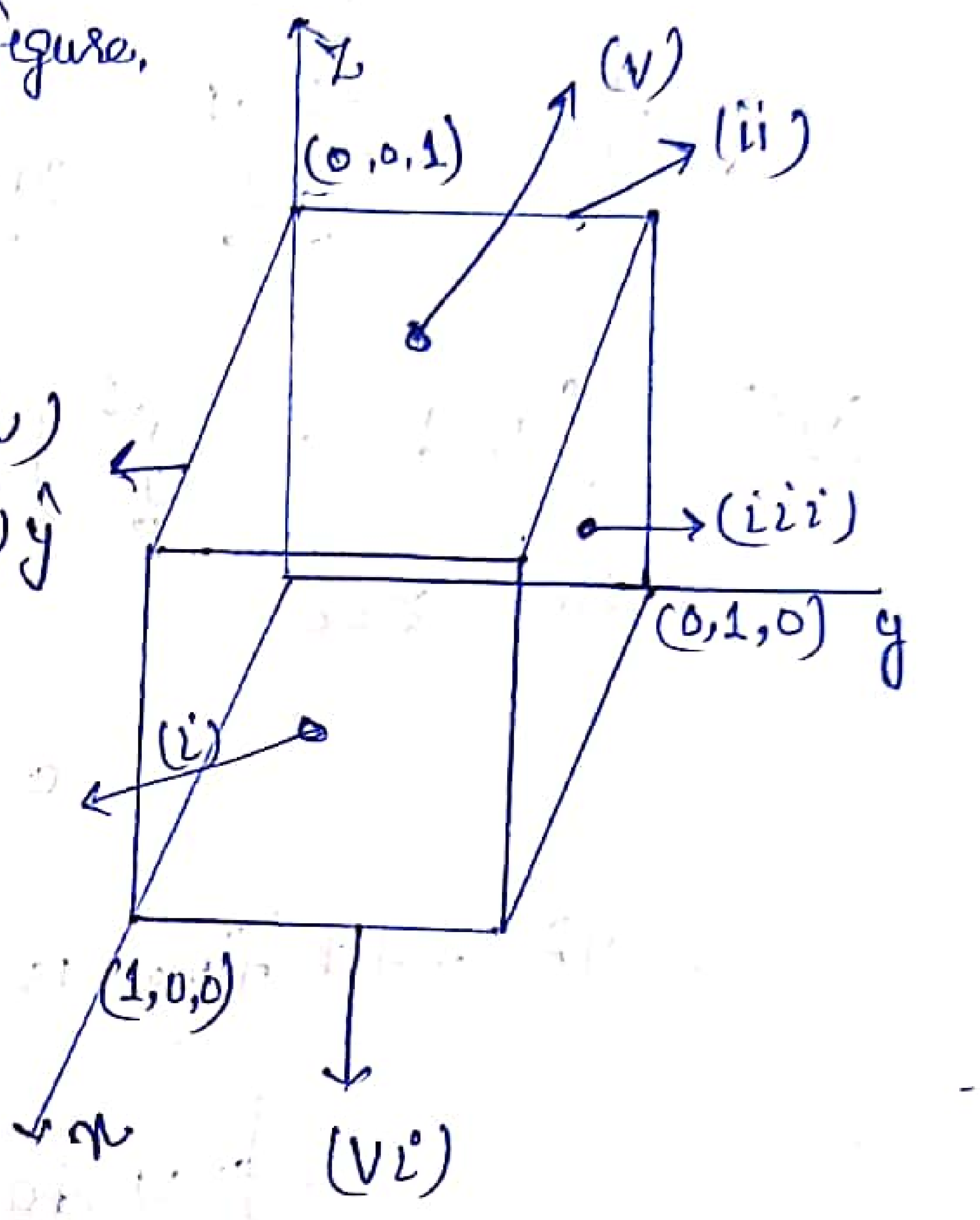
Geometrical Interpretation :-

If '$\vec{v}$' is represent the flow of incompressible fluid then the flux '$\vec{v}$' is the total amount of fluid passing out through the surface per unit time.

$$\boxed{\int \text{faucets within the Volume} = \int \text{flow out through the surface}}$$

Example :→ Check the divergence theorem using the function

$$\vec{V} = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$$ and a unit cube at the origin as shown in figure.



Part. 1 L.H.S.

~~the~~ Check divergence Theorem

div. ~~v~~ =

$$\nabla\cdot V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z}\right)$$

$$= 2x + 2y = 2(x+y)$$

$$\iiint \text{div}.v\, d\tau$$

$$= \iiint 2(x+y)\,d\tau$$

$$= 2\int_0^1\!\int_0^1\!\int_0^1 (x+y)\,dx\,dy\,dz = 2\left[\int_0^1\!\int_0^1\!\int_0^1 \{x\,dx\,dy\,dz + y\,dx\,dy\,dz\}\right]$$

$$= 2\left[\left(\frac{x^2}{2}\right)_0^1 + \left[\frac{y^2}{2}\right]_0^1\right] = 2\left\{\frac{1}{2} + \frac{1}{2}\right\} = 2$$

R.H.S → We have consider six faces of ~~cubic~~ cube

$$\text{face (i)} = \int_0^1\!\int_0^1 V\cdot da = \int_0^1\!\int_0^1 [y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z}]\cdot[\hat{x}\,dy\,dz]$$

$$= \int_0^1\!\int_0^1 y^2\,dy\,dz = 1/3$$

(ii) $$\Rightarrow \int V\cdot da = -\int_0^1\!\int_0^1 y^2\,dy\,dz = -1/3$$

(iii) $$\int V\cdot da = \int_0^1\!\int_0^1 (y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z})\cdot(\hat{y}\,dx\,dz)$$

$y=1$ $$= \int_0^1\!\int_0^1 (2xy + z^2)\,dx\,dz = \int_0^1\!\int_0^1 (2x + z^2)\,dx\,dz = \frac{4}{3}$$

(iv) $$\int v\cdot da = -\int_0^1\!\int_0^1 (2xy + z^2)\,dx\,dz = -\int_0^1\!\int_0^1 z^2\,dx\,dz = -\frac{1}{3}$$

$y=0$

(V) $\int v \cdot da = \int_0^1 \int_0^1 y^2 \; v \cdot (\hat{z}\, dx dy)$

$= \int_0^1 \int_0^1 [y^2 \hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}] \cdot \hat{z}\, dx\, dy$

$\{z=1\}$ $= \int_0^1 \int_0^1 2yz\, dx\, dy = \int_0^1 \int_0^1 2y\, dx dy = 1$

(vi) $\int v \cdot da = -\int_0^1 \int_0^1 2yz\, dx\, dy.$

Here $z=0$

$= 0$

So the total flux is

$$\oint_S v \cdot da = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0. = 2$$

R.H.S = L.H.S.

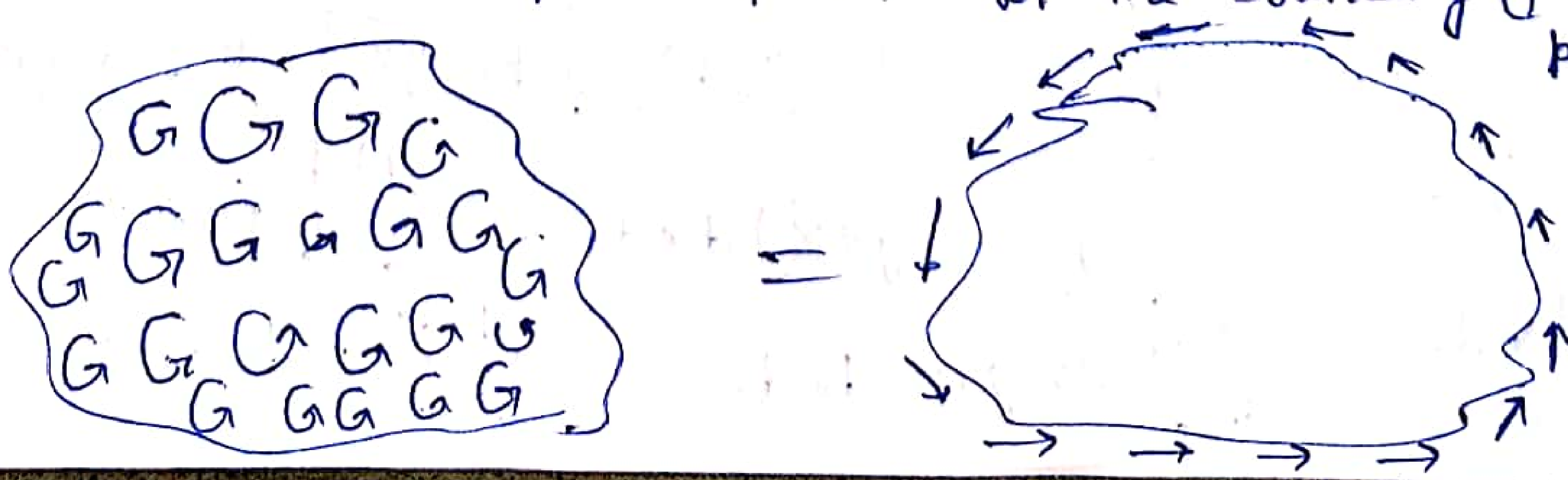
Divergence theorem is proved.

Fundamental Theorem for Curls → (Stokes Theorem)

fundamental Theorem says that

$$\iint (\nabla \times v) \cdot da = \oint v \cdot dl.$$

The integral of a derivative (the curl) over a region (a path of Surface S) is equal to the value of the function at the boundary (perimeter of patch P)



Geometrical Representation of Stokes' theorem or Fundamental Theorem ~~of~~ for curl

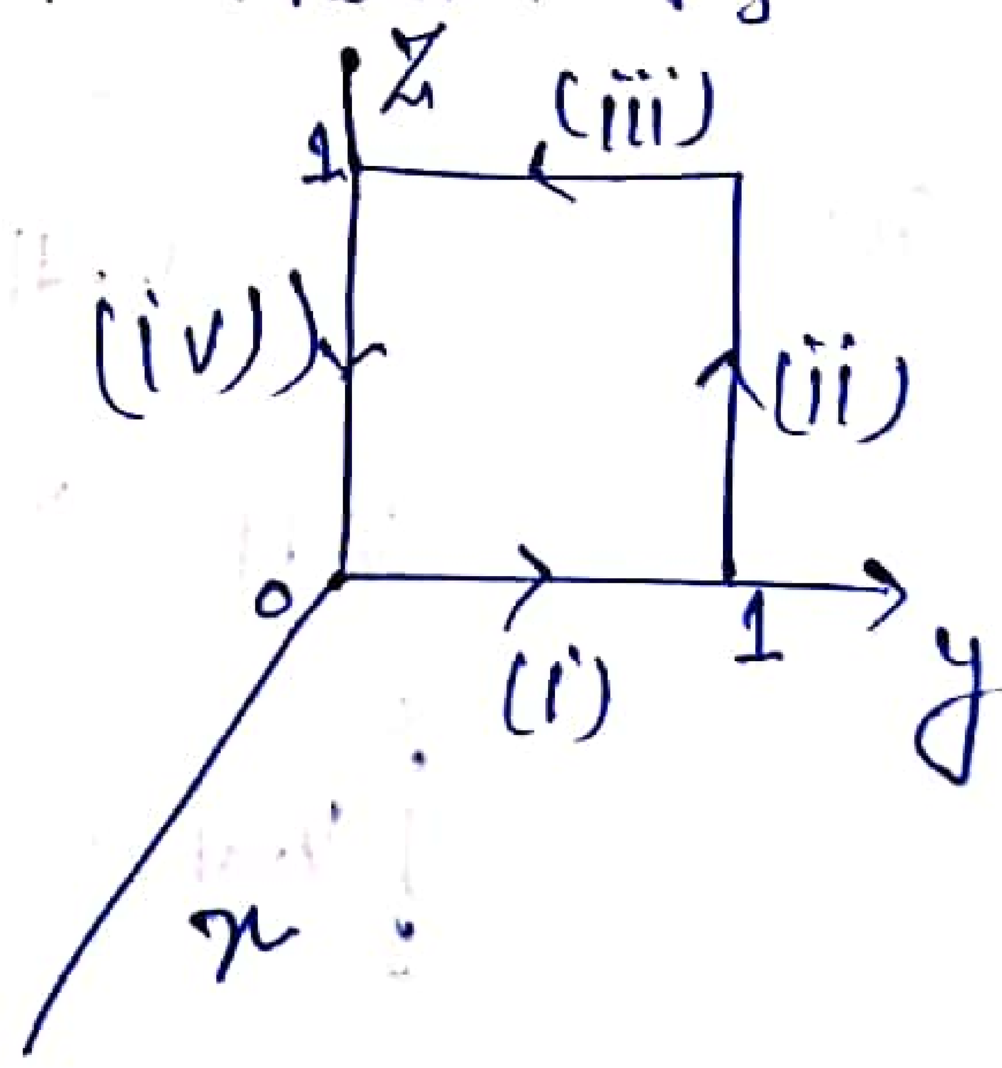
As we know that the curl measures the "twist" of the vector V; a region of high curl is a whirlpool.
If we put a tiny paddle wheel there, it will rotate. So
Now, the integral of the curl over some surface (or, more precisely, the flux of the curl through that surface) represents the "total amount of swirl" and we can determine that just as well by ~~goue~~ going around the edge and finding how much the flow is following the boundary.

$\oint V \cdot dl \rightarrow$ called circulation of V

Faraday's Law: $\rightarrow$ curl of electric field to the rate of change of a magnetic field

Example: $\rightarrow$ Suppose $V = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.

$$\boxed{\iint (\nabla \times V)\, da = \oint V \cdot dl}$$



$$\nabla \times V = \nabla \times \left((2xz + 3y^2)\hat{y} + 4yz^2\hat{z}\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix}$$

$$= \hat{x}\left|\frac{\partial}{\partial y}(4yz^2) - \frac{\partial}{\partial z}(2xz + 3y^2)\right| + \hat{y}\left|0 - \underbrace{\frac{\partial}{\partial x}(4yz^2)}_{0}\right| + \hat{z}\left|\frac{\partial}{\partial x}(2xz + 3y^2) - 0\right|$$

$$\boxed{\nabla \times V = (4z^2 - 2x)\hat{x} + 2z\hat{z}}$$

$da = \hat{x}\, dy\, dz$

$$\iint (\nabla \times V)\cdot da = \int_0^1 \int_0^1 [(4z^2 - 2x)\hat{x} + 2z\hat{z}]\cdot\hat{x}\, dy\, dz$$

$x = 0$
for this surface $= \int_0^1 \int_0^1 4z^2\, dy\, dz = \frac{4}{3}$.

Now, what about the line Integral? We must break this up in into four segments

(i) $x = 0, z = 0$ $\quad V\cdot dl = [(2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}]\cdot dy\,\hat{y}$

$$= 3y^2 dy$$

$$\int_0^1 V\cdot dl = \int_0^1 3y^2 dy = 1$$

(ii) $x = 0, y = 1$ $\quad V\cdot dl = [(\underline{2xz + 3y^2})\hat{y} + (4yz^2)\hat{z}]\cdot\hat{z}\, dz$

$$= 4yz^2 dz = 4z^2 dz$$

$$\int_0^1 V\cdot dl = \int_0^1 4z^2 dz = \frac{4}{3}$$

(iii) $x = 0, z = 1$ $\quad V\cdot dl = [\underline{(2xz + 3y^2)}\hat{y} + (4yz^2)\hat{z}]\cdot\hat{y}\, dy$

$$V\cdot dl = 3y^2 dy$$

$$\int_1^0 V\cdot dl = \int_1^0 3y^2 dy = 3\left(-\frac{1}{3}\right) = -1$$

(iv) $x = 0, y = 0$ $\quad V\cdot dl = [\underline{(2xz + 3y^2)}\hat{y} + 4yz^2\hat{z}]\cdot\hat{z}\, dz$

$$= 4yz^2 dz. \quad = 0$$

$$\int_1^0 V\cdot dl = \int_1^0 0 = 0$$

Total Sum $\oint V\cdot dl = 1 + 4/3 - 1 + 0 = 4/3$

Hence proved $\quad$ Stokes' Theorem is proved.